EQUIVALENCE OF GROUP ACTIONS ON RIEMANN SURFACES

MARIELA CARVACHO BUSTAMANTE

Abstract. We produce for each natural number $n \geq 3$ two 1–parameter families of Riemann surfaces admitting automorphism groups with two cyclic subgroups H_1 and H_2 of orden 2^n, that are conjugate in the group of orientation–preserving homeomorphism of the corresponding Riemann surfaces, but not conjugate in the group of conformal automorphisms.
This property implies that the subvariety $\mathcal{M}_g(H_1)$ of the moduli space $\mathcal{M}_g$ consisting of the points representing the Riemann surfaces of genus g admitting a group of automorphisms topologically conjugate to H_1 (equivalently to H_2) is not a normal subvariety.

1. Introduction

When we consider a group G and say that G *acts on a Riemann surface* S, we are saying that there exists a group monomorphism from G to $\mathrm{Aut}(S)$, where $\mathrm{Aut}(S)$ is the group consisting of the self–maps of S (automorphism or bi–holomorphic map) which preserve the complex structure.

Two subgroups, say $H_0\,, H_1 < Aut(S)$ are said to be *conformally equivalent* (respectively, *topologically equivalent*) if there exists an automorphism (respectively, an homeomorphism) $t : S \to S$ so that $tH_0t^{-1} = H_1$.
It is clear from the definition that any two conformally equivalent subgroups are topologically equivalent, but the reciprocal is in general false.
G. González in [5, 6] proved that if H_0 and H_1 are cyclic groups of order p prime, S/H_0 is the Riemann sphere and H_0 and H_1 are topologically equivalent, they should be conformally equivalent.
Continuing with the cyclic case for a group of prime order, a relationship between two topologically equivalent actions for the generating vectors is given by J. Gilman in [9]. The case where the group is cyclic, a relationship between the local structure for the automorphisms with fixed points and the epimorphism associated to the action is given by W. Harvey in [10].
Later in [8], G. González–Diez and R. Hidalgo give an example of two actions of $\mathbb{Z}/8\mathbb{Z}$ on a family of compact Riemann surfaces of genus 9 that are directly topologically, but not conformally, equivalent, except for finitely many cases.

Studying the classification of actions contributes to the understanding of the properties of the moduli space $\mathcal{M}_g$.
For a compact Riemann surface S_0 of genus g, consider the subgroup $H_0 \leq \mathrm{Aut}(S_0)$, the set

$$X(S_0, H_0) = \left\{(S, H) : \exists t \in \text{ Homeo }^+(S_0, S)\,, tH_0t^{-1} = H\right\}$$

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and the equivalence relation: $(S_1, H_1) \sim (S_2, H_2)$ if and only if there is $\phi \in \mathrm{Isom}(S_1, S_2)$ so that $\phi H_1 \phi^{-1} = H_2$.
We denote by $\widetilde{\mathcal{M}}_g(H_0)$ the quotient space defined by the above relation. This turns out to be a normal space.
Consider $\mathcal{M}_g$ the moduli space associated to S_0, that is, a model of moduli space of genus g.
Let $\mathcal{M}_g(H_0) = \{[S] \in \mathcal{M}_g \, : \exists t \in \text{ Homeo }^+(S_0, S)\, , tH_0t^{-1} < \mathrm{Aut}(S)\}$.
The forgetful map is defined by

$$\begin{array}{rcl} \mathcal{P} : \widetilde{\mathcal{M}}_g(H_0) & \longrightarrow & \mathcal{M}_g(H_0) \\ {[(S, H)]} & \rightsquigarrow & [S] \end{array}$$

As is well known, $\widetilde{\mathcal{M}}_g(H_0)$ is the normalization of $\mathcal{M}_g(H_0)$. Moreover, $\mathcal{P}$ is not bijective if only if there exists a compact Riemann surface S of genus g admitting two groups of automorphisms H_1 and H_2 which are directly topologically, but not conformally, conjugate to H_0. For further details, see [7].

Section 2 contains an overview of definitions and relevant results about automorphisms of Riemann surfaces and Fuchsian groups.

Section 3 contains some our contribution to the problem of the classification of actions. For cyclic groups, Theorem 3.2 gives a condition on the generating vectors under which two actions are directly topologically equivalent. Also, we generalize a result due to Harvey [10, Theorem 7].

Section 4, inspired by the paper of G. González–Diez and R. Hidalgo [8], we produce for each $n \in \mathbb{N}$ the families $\mathfrak{S}_1$ and $\mathfrak{S}_2$. By definition, $\mathfrak{S}_i$ with $i = 1, 2$, consists of the Riemann surfaces of genus $3(2^n - 1)$ defined by

$$f_{a,\lambda}\,(x, y) = y^{2^n} - x^a \left(x^2 - 1\right)^a \left(x^2 - \lambda^2\right) \left(x^2 - \lambda^{-2}\right)$$

When $i = 1$ (resp. $i = 2$), the automorphism group for the elements of $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$) is $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$). In both cases, there exist two cyclic subgroups which define directly topologically, but not conformally, equivalent actions.

2. Preliminaries

We say σ is a *holomorphic map of Riemann Surfaces* from S_1 to S_2 if, for each $P \in S_1$, (U, ϕ) chart centered at P and (V, ψ) chart centered at $Q = \sigma(Q) \in S_2$, then we have $\psi \circ \sigma \circ \phi^{-1}$ is a holomorphic function. The order at zero for this holomorphic function it is called the *multiplicity* at P. When the multiplicity at P is greater than or equal 2 we say P is a *ramification point of* σ. The point $Q = \sigma(P)$ is called *branch point of* σ.

For a bijective holomorphic map, $\sigma : S_1 \longrightarrow S_2$, we say σ is a *bi-holomorphic map* or a *isomorphism* between Riemann surfaces. When $S_1 = S_2$ we say σ is an *automorphism of* S. From now on, $\mathrm{Isom}(S_1, S_2)$ (respectively $\mathrm{Aut}(S)$) denotes the isomorphisms set between S_1 and S_2 (respectively automorphism of S).
Let S be a compact Riemann surface of genus $g \geq 2$. For each $P \in S$, consider the subgroup of $\mathrm{Aut}(S)$ given by

$$\mathrm{Aut}(S)_P = \{\sigma \in \mathrm{Aut}(S) : \sigma(P) = P\}\ ,$$

called *stabilizer subgroup.*

Now let (U, ϕ) be a chart centered at P, and $\sigma \in \text{Aut}(S)_P$ then we have

$$\phi \circ \sigma \circ \phi^{-1}(z) = \sum_{m \geq 1} c_m(\sigma) z^m$$

and we define

$$\begin{array}{rcl} \delta_P : \text{Aut}(S)_P & \longrightarrow & S^1 = \{z \in \mathbb{C} : |z| = 1\} \\ \sigma & \rightsquigarrow & c_1(\sigma) \end{array}$$

Theorem 2.1. *The map δ_P is a group monomorphism. Further,* $\text{Aut}(S)_P$ *is a cyclic finite subgroup of* $\text{Aut}(S)$.

The proof of this theorem can be found in [16].
Note that for $\tau \in \text{Aut}(S)$ and $\sigma \in \text{Aut}(S)_P$ we have $\tau \circ \sigma \circ \tau^{-1} \, \text{Aut}(S)_q$ where $Q = \tau(P)$. It is not difficult prove that $\delta_Q(\tau \circ \sigma \circ \tau^{-1}) = \delta_P(\sigma)$.

2.1. **Fuchsian groups.** Let Δ denote the unit disk $\{z : |z| < 1\}$ and let $\text{Aut}(\Delta)$ be the group of Möbius transformations self–mappings of Δ. A *Fuchsian group* is a discrete subgroup Γ of $\text{Aut}(\Delta)$. Let $PSL(2, \mathbb{C})$ denote the Möbius transformations group and let T be a Möbius transformation. If $T \neq 1$ then T has one or two fixed point. If T has one fixed point, it is called *parabolic* transformation. Now let Υ be a subgroup of $PSL(2, \mathbb{C})$. We say that Υ acts *properly discontinuously* at $z_0 \in \widehat{\mathbb{C}}$ provided that the stabilizer subgroup of Υ at z_0, Υ_{z_0}, is finite, and there exists a neighborhood U of z_0 such that

$$T(U) = U \qquad ,\forall T \in \Upsilon_{z_0} \qquad \text{and} \qquad U \cap T(U) = \emptyset \qquad ,\forall T \in \Upsilon - \Upsilon_{z_0}$$

Denote by $\Omega(\Upsilon)$ the *region of discontinuity* of Υ, that is, the set of point $z_0 \in \widehat{\mathbb{C}}$ such that Υ acts properly discontinuously at z_0. The complement of set $\Omega(\Upsilon)$ is denote by $\Lambda(\Upsilon)$ and call the *limit set* of Υ.
A Fuchsian group Γ (acting on the unit disk Δ) must satisfies that $\Delta \subset \Omega(\Gamma)$.

Let Γ be a Fuchsian group. Since Γ acts on Δ, we may consider the natural projection

$$\pi_\Gamma : \Delta \longrightarrow \Delta/\Gamma$$

$\mathcal{O} = \Delta/\Gamma$ has a *Riemann orbifold structure*, that is,

(i) an underlying Riemann surface structure $\mathcal{O}$ so that $\pi_\Gamma : \Delta \to \mathcal{O}$ is a holomorphic map;
(ii) a discrete collection of cone points (branch points of π); and
(iii) at each cone point p a cone order; this being the order of the stabilizer cyclic subgroup of any point q so that $\pi(q) = p$.

If Γ is torsion–free such that $\Lambda(\Gamma) = S^1$ then $\Pi_1(\mathcal{O}) \cong \Gamma$, and $\mathcal{O}$ has not cone point. Furthermore let Γ' be a Fuchsian groups such that Γ' is torsion–free and $\Lambda(\Gamma') = S^1$. Then $S = \Delta/\Gamma$ and $S' = \Delta/\Gamma'$ are isomorphic Riemann surfaces if only if there exists $T \in \text{Aut}(\Delta)$ such that $\Gamma' = T\Gamma T^{-1}$.
If Γ is finitely generated, without parabolic transformations and $\Lambda(\Gamma) = S^1$, then $\mathcal{O}$ is a compact Riemann surface of some genus γ and there are a finite set of cone points. Furthermore if Γ torsion–free then the genus of Δ/Γ is at less 2.
If Γ is finitely generated, without parabolic transformations and $\Lambda(\Gamma) = S^1$, whose underlying Riemann surface has genus γ and the cone orders are $m_1, \cdots, m_r$, then we define its *signature* (for both, Γ and $\mathcal{O}$) as the tuple $(\gamma; m_1, \cdots, m_r)$.

The holomorphic map π_Γ is called a *branched covering* of type $(\gamma; m_1, \cdots, m_r)$.
When a Fuchsian group Γ has signature $(\gamma; m_1, \cdots, m_r)$, there is a presentation associated for the group Γ, this is, there exist $a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r \in \Gamma$ such that Γ has a presentation:

$$(2.1) \quad \Gamma = \left\langle a_1, b_1, .., a_\gamma, b_\gamma, x_1, .., x_r : x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1 \right\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.
Further, we have that for each j, the subgroup generate for $< x_j >$ is a maximal finite cyclic subgroup. Moreover, this subgroup is the $\Gamma-$ stabilizer of a unique point in Δ, and each element of finite order in Γ is conjugate to a power of some x_j.
When x_j is conjugate (in $\mathrm{Aut}(\Delta)$) to the rotation $R(z) = \exp\left(\frac{2\pi i}{m_j}\right) z$ (respectively $R(z) = \exp\left(-\frac{2\pi i}{m_j}\right) z$), say x_j is a *positive minimal rotation* (respectively *non–positive minimal rotation*). We say that x_j is a *minimal rotation* in any these cases.
According to the paper of L.Keen [12] (also see [11, Theorem 4.3.2]) we may construct for Γ a hyperbolic polygon with $4\gamma + 2r$ sides.
Associated to this hyperbolic polygon we may find a presentation for Γ as (2.1) such that for each $j = 1, .., r$ x_j is a positive minimal rotation.
We remark that for every signature $(\gamma; m_1, m_2, \cdots, m_r)$ such that

$$2\gamma - 2 + \sum \left(1 - \frac{1}{m_j}\right) > 0$$

there exists Γ a Fuchsian group, uniquely determined up conjugation in $\mathrm{Aut}(\Delta)$, with this signature.
For more details see [14], [4], [11] and [15].

Consider two Fuchsian groups Γ_1, Γ_2 . We say that Γ_1 is *geometrically isomorphic* to Γ_2 if there exists a self–homeomorphism of Δ, say $T \in$ Homeo (Δ), and a group isomorphism $\chi : \Gamma_1 \to \Gamma_2$ such that for all $x \in \Gamma_1$ the following holds

$$\chi(x) = T \circ x \circ T^{-1}.$$

We also say that the group isomorphism $\chi : \Gamma_1 \to \Gamma_2$ can be *realized geometrically* if there exists $T \in$ Homeo (Δ) such that the previous condition is true.

Theorem 2.2. *Let $\chi : \Gamma_1 \longrightarrow \Gamma_2$ be an isomorphism between finitely generated Fuchsian group, both without parabolic elements, with $\Lambda(\Gamma_j) = S^1$, for $j = 1, 2$. Then χ is geometric.*

The preceding theorem holds at the level of Non Euclidean plane Crystallographic groups (NEC groups), that is, finitely generated discrete subgroup of isometries of the hyperbolic disc containing no parabolic elements. For more details see [14, pag. 1201].
At this point it is important to note that two homeomorphism, say $F_1, F_2 : \Delta \longrightarrow \Delta$, defining the same isomorphism $\chi : \Gamma_1 \longrightarrow \Gamma_2$, must have the same orientability type. In fact, the homeomorphism $F_2^{-1} \circ F_1 : \Delta \longrightarrow \Delta$ defines the identity automorphism of Γ_1. It can be proved that, in this case, $F_2^{-1} \circ F_1$ is homotopic to the identity.

We remark that in general we may not assume the homeomorphism that realizes the isomorphism should be orientation preserving. An example of this situation is: let Γ be a Fuchsian group with signature $(0; 5, 5, 5)$ the isomorphism $\chi : \Gamma \to \Gamma$ given by $\chi(x_j) = x_j^{-1}$ (for $j = 1, 2$ and notations as (2.1)).

2.2. **Actions on Riemann surfaces.** Let $\varepsilon : G \to \mathrm{Aut}(S)$ be an action of G on S, where S is a Riemann surfaces of genus at less 2 and G is a finite group.
We may consider for each $P \in S$ the *stabilizer subgroup* of P, this is

$$G_P = \{D \in G : \varepsilon(D)(P) = P\} \ .$$

Since $\varepsilon(G_p) \leq \mathrm{Aut}(S)_P$ it follow of the theorem (2.1) that G_P is a cyclic group.
For the action $\varepsilon : G \to \mathrm{Aut}(S)$ we have the natural projection $\pi : S \longrightarrow S/\varepsilon(G)$. Using this projection we can give to $S/\varepsilon(G)$ an complex structure. Hence $S/\varepsilon(G)$ is a compact Riemann surfaces and π is a holomorphic map of Riemann surfaces. Further we have that $P \in S$ is a ramification point of π if only if $G_P \neq \{Id\}$, furthermore the multiplicity of P is $|G_P|$. Then π is a smooth covering (unbranched covering) on the complement of a finite set, the ramification points set.
We called to π a *branched covering*, and we say that π has a *signature* $(\gamma; m_1, \cdots, m_r)$, where γ is the genus of $S/\varepsilon(G)$, m_j are the multiplicity of the ramification points, and r is the number of the branch points of π. Sometimes also we will say G acts on S with signature $(\gamma; m_1, \cdots, m_r)$. For this notation we suppose $m_1 \geq \cdots \geq m_r$.
The following theorem give a relation between the action on Riemann surfaces theory and the Fuchsian group theory:

Theorem 2.3. *Let S be a compact Riemann surface of genus $g \geq 2$ and let G be a finite group. There is an action ε of G on S with signature $(\gamma; m_1, \cdots, m_r)$ if and only if there are a Fuchsian group Γ with signature $(\gamma; m_1, \cdots, m_r)$, an epimorphism $\theta_\varepsilon : \Gamma \to G$ such that $K = \ker(\theta_\varepsilon)$ is torsion–free Fuchsian group, $\Lambda(K) = S^1$ and Δ/K is a Riemann surface isomorphism to S.*

We have Δ/K is a Riemann surface isomorphic to S. Call $f : \Delta/K \to S$ to this isomorphism and $\pi_K : \Delta \to \Delta/K$ to the natural projection. If X is the universal covering for S, then we may lift f to $F : \Delta \to X$ isomorphism of Riemann surfaces. For $x \in \Gamma$ we define $\theta_\varepsilon(x) = f \circ \widetilde{x} \circ f^{-1}$ where $\widetilde{x}$ is the automorphism of Δ/K induced by x (i.e. $\pi_K \circ x = \widetilde{x} \circ \pi_K$).

3. Equivalence of actions

Let S_j be a Riemann Surface for $j = 1, 2$, and G be a group. The actions ε_1, ε_2 of G on S_1 and S_2 respectively, are called *topologically equivalent* if there exist $\Phi \in \mathrm{Aut}\,(G)$ and $t \in$ Homeo (S_1, S_2) such that the following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon_1} & \varepsilon_1(G) \\ \Phi \downarrow & & \downarrow \Psi_t \\ G & \xrightarrow{\varepsilon_2} & \varepsilon_2(G) \end{array}$$

where $\Psi_t(\tau) := t \circ \tau \circ t^{-1}$ and Homeo (S_1, S_2) is the group of homeomorphisms of S_1 on S_2.

If $t \in$ Homeo $^+(S_1, S_2)$ then we say that ε_1, ε_2 are *directly topologically equivalent*, where Homeo $^+(S_1, S_2)$ is the group of orientation preserving homeomorphisms of S_1 on S_2. Further if $t \in \mathrm{Isom}(S_1, S_2)$ then we say that ε_1, ε_2 are *conformally equivalent*.

If ε_j is an action of G on S_j, then by the theorem 2.3 there exist Γ_j and K_j Fuchsian groups and an epimorphism $\theta_{\varepsilon_j} = \theta_j : \Gamma_j \to G$ such that $K_j = \ker(\theta_j)$. The following theorem we give a relation between θ_1 and θ_2 when the actions ε_1 and ε_2 are equivalents of some type.

Theorem 3.1. *ε_1 is topologically equivalent to ε_2 (respectively directly topologically equivalent or conformally equivalent) if only if there exists $T \in$ Homeo (Δ) (respectively $T \in$ Homeo $^+(\Delta)$ or $T \in \mathrm{Aut}(\Delta)$) and a group isomorphism $\Phi : \varepsilon_1(G) \to \varepsilon_2(G)$ such that $\Phi \circ \theta_1 = \theta_2 \circ \chi_T$ where $\chi_T(x) = T \circ x \circ T^{-1}$.*

Proof. For each j, call $f_j : \Delta/K_j \to S_j$ to the isomorphism between Δ/K_j and S_j given by the theorem 2.3. If ε_1 is topologically equivalent to ε_2 (respectively directly topologically equivalent or conformally equivalent) then there exists $t \in$ Homeo (S_1, S_2) (respectively $t \in$ Homeo $^+(S_1, S_2)$ or $t \in \mathrm{Isom}(S_1, S_2)$). Now we may lift $f_2^{-1} \circ t \circ f_1$ to $T \in$ Homeo (Δ) (respectively $T \in$ Homeo $^+(\Delta)$ or $T \in \mathrm{Aut}(\Delta)$). For $x \in \Gamma_1$, according to the notation of the theorem 2.3, we have the following diagram:

$$\begin{array}{ccccccc}
\Delta & \xrightarrow{T^{-1}} & \Delta & \xrightarrow{x} & \Delta & \xrightarrow{T} & \Delta \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Delta/K_2 & \xrightarrow{f_1^{-1} t^{-1} f_2} & \Delta/K_1 & \xrightarrow{f_1^{-1} \theta_1(x) f_1} & \Delta/K_1 & \xrightarrow{f_2^{-1} t f_1} & \Delta/K_2
\end{array}$$

Note that $T \circ x \circ T^{-1} \in \Gamma_2$. Furthermore if $x \in K_1$ then $T \circ x \circ T^{-1} \in K_2$. Using the diagram we have $\theta_2(T \circ x \circ T^{-1}) = \Psi_t(\theta_1(x))$.

Reciprocally we have $\Phi \circ \theta_1 = \theta_2 \circ \chi_T$ then $\chi_T(K_1) = K_2$. Therefore $T \in$ Homeo (Δ) (respectively $T \in$ Homeo $^+(\Delta)$ or $T \in \mathrm{Aut}(\Delta)$) define $t \in$ Homeo (S_1, S_2) (respectively $t \in$ Homeo $^+(S_1, S_2)$ or $t \in \mathrm{Isom}(S_1, S_2)$) such that $\Phi = \Psi_t$ and the actions are topologically equivalents (respectively directly topologically equivalent or conformally equivalent). □

Remark that for $S_1 = S_2 = S$ the actions ε_1 and ε_2 are topologically equivalents (respectively directly topologically equivalent or conformally equivalent) if only if $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are conjugate group in Homeo (S) (respectively in Homeo $^+(S)$ or $\mathrm{Aut}(S)$).

3.1. **Cyclic groups.** Let $G = \mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. We consider G as the integers module n ($G = \{0, 1, \cdots, n-1\}$) and Γ a Fuchsian group with signature $(\gamma; m_1, \cdots, m_r)$ and presentation as (2.1).
The following lemma was given by Kuribayashi (see [13, Lemma 3.1]).

Lemma 1. *Let $\theta : \Gamma \longrightarrow G$ be a group epimorphism and assume that ν is a generator of G. Then for any permutation μ of $\{1, \cdots, r\}$ with $m_{\mu(j)} = m_j$ ($j = 1, \cdots, r$), there exists an automorphism χ of Γ such that*

(i) $\theta \circ \chi(a_i) = \theta \circ \chi(b_i) = \nu$ *with* $i = 1, \cdots, \gamma$;
(ii) $\chi(x_j) = D_j x_{\mu(j)} D_j^{-1}$, *for some* $D_j \in \Gamma$ *with* $j = 1, \cdots, r$.

Remark that in the article of Kuribayashi the automorphisms of Γ used to computed the automorphism χ are geometric induced by orientation preserving homeomorphism.
Now on, no loss of generality consider the group epimorphism θ given by

$$\begin{aligned} \theta : \Gamma &\longrightarrow G \\ a_i &\rightsquigarrow 1 \quad , \text{ with } i = 1, .., \gamma \\ b_i &\rightsquigarrow 1 \quad , \text{ with } i = 1, .., \gamma \\ x_j &\rightsquigarrow \xi_j \quad , \text{ with } j = 1, .., r \end{aligned} \tag{3.1}$$

where $K = \ker(\theta)$ is a torsion free group.
Hence, it follow

Theorem 3.2. *Let* Γ, Γ' *Fuchsian groups both with signature* $(\gamma; m_1, \cdots, m_r)$ *and presentation as* (2.1) *according to your hyperbolic polygon associated. Consider the group epimorphisms* θ, θ' *as* (3.1). *If there exists an* $s \in \mathbb{Z}$ *with* $(s, n) = 1$, *such that*

$$(\xi_1', ..., \xi_r') \equiv s(\xi_1, ..., \xi_r), \quad \mod n \tag{3.2}$$

then the actions induced by θ *on* Δ/K *and by* θ' *on* Δ/K' *are directly topologically equivalent.*

Proof. Suppose that we have the equation (3.2), then we may define the homomorphism $\Phi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$, given by $\Phi(1) = s$. Then for each j we have

$$\Phi(\xi_j) = s\xi_j = \xi_j' .$$

Since s and n are relative primes, Φ is an automorphism.
We consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \vdots & & \vdots\, \chi & & \downarrow \Phi & & \\ 0 & \longrightarrow & K' & \longrightarrow & \Gamma' & \xrightarrow{\theta'} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

hence we may define

$$\chi(a_j) = a_i', \qquad \chi(b_i) = b_j', \qquad \chi(x_j) = x_j'$$

and then we have that the diagram is commutative.
Since χ maps generators on generators, and these elements satisfy the same relation, we have χ is an isomorphism. Further by the commutative diagram $\chi(K) = K'$ therefore $\chi|_K : K \to K'$ is an isomorphism.
It follow of Theorem 3.1 that the actions induced by θ and by θ' are topologically equivalent.
Our next claim is that the actions induced by θ and θ' are directly topologically equivalent. We have to construct according to [12] the hyperbolic polygons associated Γ and Γ'.
Since x_j and x_j' are positive minimal rotation in the same angle, we have the automorphism χ is induced by a $f \in$ Homeo $^+(\Delta)$. □

Reciprocally consider ε and ε' actions on S and S', respectively, such that the signature for the actions is $(\gamma; m_1, \cdots, m_r)$. If ε is topologically equivalent to ε' then (according to the theorem 3.1) there exists $T \in$ Homeo (Δ) such that $\Psi_t \circ \theta_\varepsilon = \theta_{\varepsilon'} \circ \chi_T$. Call $\theta = \varepsilon^{-1} \circ \theta_\varepsilon$ and $\theta' = (\varepsilon')^{-1} \circ \theta_{\varepsilon'}$. Then there exists $\Phi \in \mathrm{Aut}\,(\mathbb{Z}/n\mathbb{Z})$ such that $\Phi \circ \theta = \theta' \circ \chi_T$. Therefore $\theta'\,(\chi_T(x_j)) = \Phi(1)\theta(x_j)$.
Observe that this result is a generalization of a result of J. Gilman in which n is a prime number (See [9, Lemma 2, p. 54].)

3.2. **Epimorphism and local structure.** In [10] Harvey gives a relation between cyclic covering of Riemann sphere and the epimorphisms of Fuchsian group using the rotation angles. Our result gives a relation for any covering of Riemann sphere. As we have seen, for an action of a finite group G on a compact Riemann surface S (no loss generality consider $G < \mathrm{Aut}(S)$), according to theorem 2.3 we have an epimorphism $\theta : \Gamma \longrightarrow G$, defined by $\theta(x) = f \circ \widetilde{x} \circ f^{-1}$. Furthermore $K = \ker(\theta)$ is a torsion-free Fuchsian group and Γ is a Fuchsian group with the same signature that the action, and S is isomorphic to Δ/K.
The following theorem we give a relation between the epimorphism θ, and the homomorphism δ_P , for P fixed point of the action.

Theorem 3.3. *Let $\sigma \in \mathrm{Aut}(S)_P \leq \mathrm{Aut}(S)$ of order n, and let*

$$\mathcal{L}_P = \left\{x \in \mathrm{Aut}(\Delta) : \exists z_0 \in \Delta\,, f \circ \pi_K(z_0) = P\,, x(z_0) = z_0 \text{ and } (f^{-1} \circ \sigma \circ f) \circ \pi_K = \pi_K \circ x\right\}$$

Then there is unique primitive complex nth root of unity ζ such that for all $x \in \mathcal{L}$ we have that x is conjugate to multiplication by ζ, $R(z) = \zeta z$, in $\mathrm{Aut}(\Delta)$*. Furthermore $\zeta = \delta_P(\sigma)$.*

The proof of this theorem can be found in [2].
Since ζ is a primitive complex nth root of unity, we may write $\zeta = \omega_n^j$ where $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$ and the numbers j and n are relative primes ($(j, n) = 1$). We call $\frac{2\pi j i}{n}$ the *rotation angle* for σ at P.

Now we consider Γ with presentation as (2.1) according to your hyperbolic polygon associated. Recall x_j is a positive minimal rotation. Consider z_j the fixed point of x_j. Then $x_j \in \mathcal{L}_{P_j}$ where $P_j = f(\pi_K(z_j))$.
Furthermore by the preceding theorem we have $\delta_{P_j}(\theta(x_j)) = \omega_{m_j}$.

Theorem 3.4. *Let G be a subgroup of* $\mathrm{Aut}(S)$*, where G acts on S with signature $(0; m_1, \cdots, m_r)$. The epimorphism θ is determined by the fixed points of the action and their stabilizer groups. In other words, if we consider x_j and P_j as before, then*

$$\theta(x_j) = \tau_j^{\xi_j}$$

where $<\tau_j>= G_{P_j}$, and where the number ξ_j is determined by the equations

$$\begin{aligned} \delta_{P_j}(\tau_j) &= \omega_{m_j}^{\eta_j}\,, \qquad 1 \leq \eta_j < m_j, (\eta_j, m_j) = 1, \\ \eta_j \cdot \xi_j &\equiv 1 \mod m_j \end{aligned}$$

where $1 \leq \xi_j < m_j$, with $(\xi_j, m_j) = 1$.

Proof. Recall that for the fixed point P_j we have G_{P_j} is a cyclic group of order m_j. Let τ_j be a generator of G_{P_j}. Then $\theta(x_j) = \tau_j^{\xi_j}$ for some $0 < \xi_j < m_j$ this is because $\theta(x_j) \in G_{P_j}$.

Since δ_{P_j} is a group monomorphism and $\delta_{P_j}(\theta(x_j)) = \omega_{m_j}$ then

$$\omega_j = \delta_{P_j}\left(\tau_j^{\xi_j}\right) = \delta_{P_j}(\tau_j)^{\xi_j}$$

therefore $(\xi_j, m_j) = 1$.
If $\delta_{P_j}(\tau_j) = \omega_{m_j}^{\eta_j}$ (where $(\eta_j, m_j) = 1$) then

$$\omega_{m_j} = \delta(\tau_j)^{\xi_j} = \omega_{m_j}^{\eta_j\,\xi_j}$$

hence

$$1 \equiv \eta_j\,\xi_j \mod m_j$$

If we take another generator of G_{P_j}, say $\widehat{\tau}_j$, then we may to do the same computations, hence

$$\theta(x_j) = \widehat{\tau}_j^{\widehat{\xi}_j}$$

where

$$\begin{aligned} \delta_{P_j}(\widehat{\tau}_j) &= \omega_{m_j}^{\widehat{\eta}_j} \\ \widehat{\eta}_j \cdot \widehat{\xi}_j &\equiv 1 \mod m_j \end{aligned}$$

As τ_j is a generator of G_{P_j}, we have there exists $0 < t < m_j$ such that $\widehat{\tau}_j = \tau_j^t$ thus we have

$$\omega_{m_j}^{\widehat{\eta}_j} = \delta_{P_j}(\widehat{\tau}_j) = \delta_{P_j}(\tau_j^t) = \omega_{m_j}^{\eta_j \cdot t}$$

then

$$\begin{aligned} \widehat{\eta}_j &\equiv \eta_j \cdot t \mod m_j \qquad | \cdot \xi_j \\ \widehat{\eta}_j \cdot \xi_j &\equiv \eta_j \cdot \xi_j t \equiv t \mod m_j \qquad | \cdot \widehat{\xi}_j \\ \xi_j \equiv \widehat{\xi}_j \cdot \widehat{\eta}_j \cdot \xi_j &\equiv t \cdot \widehat{\xi}_j \mod m_j \end{aligned}$$

therefore $\tau_j^{\xi_j} = \widehat{\tau}_j^{\widehat{\xi}_j}$. □

Remark that as θ is an epimorphism for the $r-$tuple $(\theta(x_1), \cdots, \theta(x_r))$ is has

(1) $G =< \theta(x_1), \cdots, \theta(x_r) >$.
(2) $\text{ord}(\theta(x_j)) = m_j$, for each $j = 1, .., r$.
(3) $\theta(x_1) \cdots \theta(x_r) = 1$.

In general we say that the $r-$tuple $(\sigma_1, \cdots, \sigma_r) \in G^r$ is a *generating vector* for G of type $(0; m_1, \cdots, m_r)$ if this satisfy the three preceding conditions replace $\theta(x_j)$ by σ_j.
Thus given the action on S we have a generating vector of type the signature of this action. The reciprocal is know as the Existence Riemann theorem. For more detail see [3].

4. Families of Riemann surfaces with equivalent actions

In this section we produce for each n, two families of Riemann surfaces of genus $3(2^n-1)$ with group of automorphisms of order 2^{n+2} and signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For one of the families the group we will be abelian, and for the other it will be a semidirect product. In both cases, we will have that there exist two cyclic subgroups which define directly topologically, but not conformally, equivalent actions.

Theorem 4.1. *Let $f_{a,\lambda}$ be the polynomial given by*

$$f_{a,\lambda}(x,y) = y^{2^n} - x^a\left(x^2-1\right)^a\left(x^2-\lambda^2\right)\left(x^2-\lambda^{-2}\right)$$

where $n, a \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $\lambda^4 \neq 1, 0$.
Then, for each odd number a and each λ, we have that $f_{a,\lambda}$ defines a Riemann surface $S_{a,\lambda}$ of genus $3(2^n-1)$.
Furthermore, the possible singular points for the homogeneous polynomial associated to $f_{a,\lambda}$ are

case	*sign*(2^n-3a-4)	a	*condition*	*singular points*
1	$+$	1	$n=3$	$\emptyset$
2	$+$	$\neq 1$	$2^n-3a-5=0$	$\{[0,0,1],[1,0,1],[-1,0,1]\}$
3	$+$	1	$2^n-3a-5\neq 0$	$\{[1,0,0]\}$
4	$+$	$\neq 1$	$2^n-3a-5\neq 0$	$\{[0,0,1],[1,0,0],[1,0,1],[-1,0,1]\}$
5	$-$	$\neq 1$		$\{[0,0,1],[0,1,0],[1,0,1],[-1,0,1]\}$
6	$-$	1	$n=1,2$	$\{[0,1,0]\}$

Table 1: Singular points

Proof. Observe that if $2^n-3a-4=0$ then a is an even number. Moreover $2^n-5-3a=0$ for some odd number a, if only if n is an odd number. Particularly if $a=1$ then $n=3$.
We have two cases for the homogeneous polynomial associated to $f_{a,\lambda}$.

(1) Case $2^n-3a-4>0$.

$$F_1(X,Y,Z) = Y^{2^n} - X^a Z^{2^n-3a-4}\left(X^2-Z^2\right)^a\left(X^2-\lambda^2Z^2\right)\left(X^2-\lambda^{-2}Z^2\right)$$

where $[X,Y,Z]\in\mathbb{P}^2\mathbb{C}$ (Projective plane) and $x=\frac{X}{Z}$, $y=\frac{Y}{Z}$.
For

- $a\neq 1, 2^n-3a-5\neq 0$, we have 4 singular points, they are

$$\{[0,0,1],[1,0,0],[1,0,1],[-1,0,1]\}$$

By the Normalization process we have the following charts for theses points

singular point	local coordinate	
$[0,0,1]$	$s \rightsquigarrow [s^{2^n}, s^a h_0(s), 1]$	$h_0(0)\neq 0$
$[1,0,1]$	$s \rightsquigarrow [s^{2^n}+1, s^a h_1(s), 1]$	$h_1(0)\neq 0$
$[-1,0,1]$	$s \rightsquigarrow [s^{2^n}-1, s^a h_{-1}(s), 1]$	$h_{-1}(0)\neq 0$
$[1,0,0]$	$t \rightsquigarrow [1, t^{2^n-3a-4}h_\infty(t), t^{2^n}]$	$h_\infty(0)\neq 0$

Table 2: Local coordinates for singular points case F_1

where for each j , h_j is an holomorphic maps defined on an open subset of complex plane.
- $a\neq 1, 2^n-3a-5=0$, then n is an odd number and the singular points are $\{[0,0,1],[1,0,1],[-1,0,1]\}$.
- $a=1, 2^n-3a-5\neq 0$, then $n>3$, since $2^n-7>0$ and $2^n-8\neq 0$. Then the singular points are $\{[1,0,0]\}$
- $a=1, 2^n-3a-5=0$, then $n=3$, since $2^n-7>0$ and $2^n-8=0$. Then in this case F_1 has not singular points.

(2) Case $2^n - 3a - 4 < 0$.

$$F_2(X,Y,Z) = Y^{2^n} Z^{3a+4-2^n} - X^a \left(X^2 - Z^2\right)^a \left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right).$$

If $a \neq 1$, we have that the singular points are

$$\{[0,0,1],[1,0,1],[-1,0,1],[0,1,0]\}$$

By the Normalization process we have the following charts for theses points

point	local coordinate	
$[0,0,1]$	$s \rightsquigarrow [s^{2^n}, s^a g_0(s), 1]$	$g_0(0) = 1$
$[1,0,1]$	$s \rightsquigarrow [s^{2^n}+1, s^a g_1(s), 1]$	$g_1(0) \neq 0$
$[-1,0,1]$	$s \rightsquigarrow [s^{2^n}-1, s^a g_{-1}(s), 1]$	$g_{-1}(0) \neq 0$
$[0,1,0]$	$t \rightsquigarrow [t^{3a+4-2^n} g_\infty(t), 1, t^{3a+4}]$	$g_\infty(0) = 1$

Table 3: Local coordinates for singular points case F_2

where for each j, g_j is an holomorphic maps defined on an open subset of complex plane.
Now if $a = 1$ as $2^n - 3a - 4 < 0$ then $n \leq 2$. These cases are studied by G. González-Diez and R. Hidalgo in [8].

Now using the Normalization process we get a Riemann Surfaces of genus $3(2^n-1)$. This is because we may define the holomorphic map $\pi : S_{a,\lambda} \longrightarrow \widehat{\mathbb{C}}$ given by $\pi(x,y) = x$.
Note that π has degree 2^n. The ramification point set of π is

$$B = \{P_0, Q_0, [\pm 1,0,1], [\pm\lambda,0,1], [\pm\lambda^{-1},0,1]\} \tag{4.1}$$

where $P_0 = [0,0,1]$ and either $Q_0 = [1,0,0]$ (if $2^n - 3a - 4 > 0$) or $Q_0 = [0,1,0]$ (if $2^n - 3a - 4 < 0$). The branch point set is $\pi(B) = \{0, \infty, \pm\lambda, \pm\lambda^{-1}, \pm 1\}$. For each $P \in B$ we have the multiplicity of P is 2^n.
By the Riemann–Hurwitz formula it follow the genus of S is $3(2^n - 1)$. □

The following theorem yields information about the automorphisms group of $S_{a,\lambda}$. The interest of the theorem is in the assertion that for each Riemann surface $S_{a,\lambda}$ the automorphisms group is not trivial (except some cases).

Theorem 4.2. *Let τ_1, τ_2 be the self–maps of $S_{a,\lambda}$ defined by*

$$\begin{aligned} \tau_1(x,y) &= (-x, \omega_{2^{n+1}} y) \\ \tau_2(x,y) &= \left(\frac{1}{x}, \frac{\omega_{2^{n+1}} y}{x^c}\right) \end{aligned}$$

where c is a natural number determined by the equation $c \cdot 2^{n-1} = 2a + 2$.
Then $\tau_1, \tau_2 \in \mathrm{Aut}(S_{a,\lambda})$ and they have order 2^{n+1} each.
Furthermore depending on the values for a we have:

(1) *If c is an even number, then $G_1 =< \tau_1, \tau_2 >$ is an abelian group, isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*
We call $\mathfrak{S}_1$ the corresponding family of surfaces.

(2) *If c is an odd number, then $G_2 =< \tau_1, \tau_2 >$ is a group isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$, where $\tau_2\tau_1 = \tau_1^{2^n+1}\tau_2$.*
We call $\mathfrak{S}_2$ the corresponding family of surfaces.

Proof. It is not difficult to check that $f_{a,\lambda}(\tau_1(x,y)) = 0$ and $f_{a,\lambda}(\tau_2(x,y)) = 0$. Note that

$$\tau_1^2(x,y) = \tau_2^2(x,y) = (x, \omega_{2^{n+1}}^2 y)$$

Then the order of τ_1^2 is 2^n, since $\omega_{2^{n+1}}^2$ is a 2^n-th primitive root of unity, therefore τ_1 and τ_2 has order 2^{n+1}.
Now we consider the homogeneous coordinates for τ_1, τ_2. They are given by

$$\begin{aligned} \tau_1[X,Y,Z] &= [-X, \omega_{2^{n+1}} Y, Z] \\ \tau_2[X,Y,Z] &= [ZX^{c-1}, \omega_{2^{n+1}} Y Z^{c-1}, X^c] \end{aligned}$$

To prove that $\tau_1, \tau_2 \in \mathrm{Aut}(S_{a,\lambda})$, we must prove that for any charts $(U,\varphi), (V,\phi)$, on $S_{a,\lambda}$, such that $\tau_j(V) \cap U \neq \emptyset$, the map $\varphi \circ \tau_j \circ \phi^{-1}$, is a holomorphic function (on some subset of $\mathbb{C}$).
We will do the computations for τ_2 at P_0 and suppose $2^n - 3a - 4 > 0$.
Using the chart at P_0 (see table 2, theorem 4.1) we have

$$\tau_2[s^{2^n}, s^a h_0(s), 1] \quad = \quad \left[1, s^{2^n - 3a - 4} \omega_{2^{n+1}} h_0(s), s^{2^n}\right] .$$

then $\tau_2(P_0) = Q_0$.
Now consider the chart at Q_0 given in the table 2, theorem 4.1. Then

$$\varphi \circ \tau_2 \circ \phi^{-1}(s) = \varphi \circ \tau_2[s^{2^n}, s^a h_0(s), 1] = \varphi[1, s^{2^n - 3a - 4} \omega_{2^{n+1}} h_0(s), s^{2^n}] = \omega s$$

where ω is a 2^n-th root of unity (we recall that $t^{2^n} = s^{2^n}$). Therefore, the map is a holomorphic function.
It is not difficult to verify the preceding process for the other charts.
Recalling that $2^{n-1}c = 2a + 2$, we have

$$\tau_1\tau_2[X,Y,Z] \quad = \quad [-ZX^{c-1}, \omega_{2^{n+1}}^2 Z^{c-1} Y, X^c]$$

$$\tau_2\tau_1[X,Y,Z] \quad = \quad \begin{cases} \tau_1\tau_2[X,Y,Z] & , c \text{ is an even number} \\ \\ \tau_1^{2^n+1}\tau_2[X,Y,Z] & , c \text{ is an odd number} \end{cases} .$$

It is not difficult compute the elements of the group generated by τ_1 and τ_2. The cardinality of this group is 2^{n+2}.
If c is an even number then the group is abelian. It is not difficult prove that the group $< \tau_1, \tau_2 >$ is isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this case we call $G_1 =< \tau_1, \tau_2 >$.
Now if c is an odd number then the group $< \tau_1, \tau_2 >$ is isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$. In fact, the element $\nu = \tau_1^{2^{n-1}-1}\tau_2$ has order 2 and we may define $h :< \nu > \to \mathrm{Aut}(< \tau_1 >)$ given by $h(\nu)(\tau_1) = \tau_1^{2n+1}$. In this case we call $G_2 =< \tau_1, \tau_2 >$. □

We remark that for $a = 1$ as $c \in \mathbb{Z}$ then $n \leq 3$. If $n < 3$ then c is an even number and theses cases were studied [8]. If $n = 3$ then $c = 1$. Furthermore we conclude that the case (3) in the table 1, theorem 4.1, it has not automorphisms of type τ_j for $j = 1, 2$.
For the case (2) in the table 1, theorem 4.1, also it has not automorphisms of type τ_j for $j = 1, 2$.

By the preceding theorem for $S_1 \in \mathfrak{S}_1$ we have that the group $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq G_1$ acts on S_1. Now we are interest in compute the signature of this action and the signature of yours subgroups. The following theorem summarize this information.

Theorem 4.3. *The cyclic subgroups of G_1 acting with fixed points (different) are given as follows:*

(1) $H_1 =< \tau_1 >$ *subgroup of order* 2^{n+1}*, acting on* S_1 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.
(2) $H_2 =< \tau_2 >$ *subgroup of order* 2^{n+1}*, acting on* S_1 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.
(3) $H_3 =< \tau_1^2 >$ *subgroup of order* 2^n*, acting on* S_1 *with signature* $(0; 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n)$.
(4) $H_4 =< \tau_1^{2^{n-1}c-1}\tau_2 >$ *subgroup of order* 2*, acting on* S_1 *with signature* $(2^n - 1; 2^{n+1} \cdot 2)$ *(*$\tau_1^{2^{n-1}c-1}\tau_2$ *has* 2^{n+1} *fixed points).*

Furthermore, we have that the group G_1 *acts on* S_1 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$ *and that* $G_1 = \mathrm{Aut}(S_1)$*, except for finitely many* $S_1 \in \mathfrak{S}_1$*.*

Proof. We will compute the fixed points.

For τ_1 the fixed points on S_1 are $\{P_0, Q_0\}$ (recall equation (4.1)).

For τ_2 the fixed points on S_1 are $\{[1, 0, 1], [-1, 0, 1]\}$. Remark that $\tau_2(P_0) = Q_0$.

For τ_1^j where j is an even number the set of fixed points on S_1 is B (recall equation (4.1)).

If $\tau^j\tau_2$ has fixed point then $j = 3 \cdot 2^{n-1}c - 1$ or $j = 2^{n-1}c - 1$. Since c is an even number then

$$\tau_1^{3\cdot 2^{n-1}c-1}\tau_2 = \tau_1^{2^{n-1}c-1}\tau_2 \tag{4.2}$$

this is because $3 \cdot 2^{n-1}c - 1 \equiv 2^{n-1}c - 1 \mod 2^{n+1}$. In this case the fixed points on S_1 are

$$\left\{ \begin{array}{ll} [i, p, 1]: & p^{2^n} = -(2i)^a(1 + \lambda^2)(1 + \lambda^{-2}) \\ [-i, q, 1]: & q^{2^n} = (2i)^a(1 + \lambda^2)(1 + \lambda^{-2}) \end{array} \right\}$$

It is not difficult to compute for each H_j the signature using Riemann Hurwitz formula.

Let Γ be a Fuchsian group with the above signature. By Singerman [17], there is no other Fuchsian group with signature of the form $(0; a, b, c, d)$ that contains it strictly. It follows that it may only be contained in a triangular signature. Hence by dimension arguments, Γ cannot be contained strictly in other subgroup as finite index subgroup except for a finite number of possibilities (up to conjugation by Möbius transformations). Therefore, the family of Riemann surfaces does not have any other automorphisms than those of G_1, except for finitely many $S_1 \in \mathfrak{S}_1$. □

We can now state the analogue of the preceding theorem for $S_2 \in \mathfrak{S}_2$.

Theorem 4.4. *The cyclic subgroups of G_2 acting with fixed points (different) are given as follows:*

(1) $H_1 =< \tau_1 >$ *subgroup of order* 2^{n+1}*, acting on* S_2 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.
(2) $H_2 =< \tau_2 >$ *subgroup of order* 2^{n+1}*, acting on* S_2 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.
(3) $H_3 =< \tau_1^2 >$ *subgroup of order* 2^n*, acting on* S_2 *with signature* $(0; 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n)$.

(4) $H_4 =< \tau_1^{3\cdot 2^{n-1}c-1}\tau_2 >$ *subgroup of order* 2*, acting on* S_2 *with signature* $(5\cdot 2^{n-2}-1; 2^n\cdot 2)$ *(*$\tau_1^{3\cdot 2^{n-1}c-1}\tau_2$ *has* 2^n *fixed points).*

(5) $H_5 =< \tau_1^{2^{n-1}c-1}\tau_2 >$ *subgroup of order* 2*, acting on* S_2 *with signature* $(5\cdot 2^{n-2}-1; 2^n\cdot 2)$ *(*$\tau_1^{2^{n-1}c-1}\tau_2$ *has* 2^n *fixed points).*
H_5 is a subgroup conjugate to H_4.

Furthermore, we have that the group G_2 acts on S_2 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$ and that $G_2 = \mathrm{Aut}(S_2)$, except for finitely many $S_2 \in \mathfrak{S}_2$.

The proof is similar to that theorem 4.3. Recall that in this case c is an odd number. Then the equation (4.2) is not true. Therefore there exist two subgroups of order 2 that it has fixed points. It is not difficult prove that these groups are conjugate. For any these case it has 2^n fixed points and the same signature.

For all P fixed point given by the theorems 4.3 and 4.4 we may compute the value of δ_P. According the remarks of the theorem 4.2 we have just 3 cases to compute. In the table 1 (theorem 4.1) they are: case (1), case (4) and case (5). Remark that the case (6) was study in [8].

Case (1). $n = 3$, $a = 1$, $c = 1$

order	τ	P	$\delta_P(\tau)$
16	τ_1	$[0,0,1]$	ω_{16}
16	τ_1	$[1,0,0]$	ω_{16}^9
16	τ_1^9	$[1,0,0]$	ω_{16}
16	τ_2	$[1,0,1]$	ω_{16}
16	τ_2	$[-1,0,1]$	ω_{16}^9
16	τ_2^9	$[-1,0,1]$	ω_{16}
8	τ_1^2	$[\lambda,0,1]$	ω_8
8	τ_1^2	$[-\lambda,0,1]$	ω_8
8	τ_1^2	$[\lambda^{-1},0,1]$	ω_8
8	τ_1^2	$[-\lambda^{-1},0,1]$	ω_8

Case (4). $2^n - 3a - 4 > 0$, $a > 1$, $2^n - 3a - 5 \neq 0$

order	τ	P	local auto.	$\delta_P(\tau)$
2^{n+1}	τ_1	$[0,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_0} s$ $j_0 a \equiv 1 \mod 2^{n+1}$ j_0 is an odd number	$\omega_{2^{n+1}}^{j_0}$
2^{n+1}	τ_1^a	$[0,0,1]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_1	$[1,0,0]$	$t \rightsquigarrow \omega_{2^{n+1}}^{j_\infty} t$ $j_\infty(-3a-4) \equiv 1 \mod 2^{n+1}$ j_∞ is an odd number	$\omega_{2^{n+1}}^{j_\infty}$
2^{n+1}	τ_1^{-3a-4}	$[1,0,0]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_1} s$ $j_1 a \equiv 1 \mod 2^{n+1}$ j_1 is an odd number	$\omega_{2^{n+1}}^{j_1}$
2^{n+1}	τ_2^a	$[1,0,1]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[-1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_{-1}} s$ $j_{-1} a \equiv 1 \mod 2^{n+1}$ j_{-1} is an odd number	$\omega_{2^{n+1}}^{j_{-1}}$ c is an even number

order	τ	P	local auto.	$\delta_P(\tau)$
2^{n+1}	τ_2^a	$[-1,0,1]$		$\omega_{2^{n+1}}$ c is an even number
2^{n+1}	τ_2	$[-1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_{-1}} s$ $j_{-1}a \equiv 2^n+1 \mod 2^{n+1}$ j_{-1} is an odd number	$\omega_{2^{n+1}}^{j_{-1}}$ c is an odd number
2^{n+1}	τ_2^a	$[-1,0,1]$		$\omega_{2^{n+1}}^{2^n+1}$ c is an odd number
2^{n+1}	$\tau_2^{a(2^n+1)}$	$[-1,0,1]$		$\omega_{2^{n+1}}$ c is an odd number
2^n	τ_1^2	$[\lambda,0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[-\lambda,0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[\lambda^{-1},0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[-\lambda^{-1},0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}

Case (5). $2^n - 3a - 4 < 0$, $a > 1$

order	τ	P	$\delta_P(\tau)$
2^{n+1}	τ_1	$[0,0,1]$	$\omega_{2^{n+1}}^{j_0}$
2^{n+1}	τ_1^a	$[0,0,1]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_1	$[0,1,0]$	$\omega_{2^{n+1}}^{j_\infty}$
2^{n+1}	τ_1^{-3a-4}	$[0,1,0]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[1,0,1]$	$\omega_{2^{n+1}}^{j_1}$
2^{n+1}	τ_2^a	$[1,0,1]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[-1,0,1]$	$\omega_{2_{n+1}}^{j_{-1}}$ c is an even number
2^{n+1}	τ_2^a	$[-1,0,1]$	$\omega_{2_{n+1}}$ c is an even number
2^{n+1}	τ_2	$[-1,0,1]$	$\omega_{2^{n+1}}^{j_{-1}}$ c is an odd number
2^{n+1}	τ_2^a	$[-1,0,1]$	$\omega_{2_{n+1}}^{2^n+1}$ c is an odd number
2^{n+1}	$\tau_2^{a(2^n+1)}$	$[-1,0,1]$	$\omega_{2_{n+1}}$ c is an odd number
2^n	τ_1^2	$[\lambda,0,1]$	ω_{2^n}
2^n	τ_1^2	$[-\lambda,0,1]$	ω_{2^n}
2^n	τ_1^2	$[\lambda^{-1},0,1]$	ω_{2^n}
2^n	τ_1^2	$[-\lambda^{-1},0,1]$	ω_{2^n}

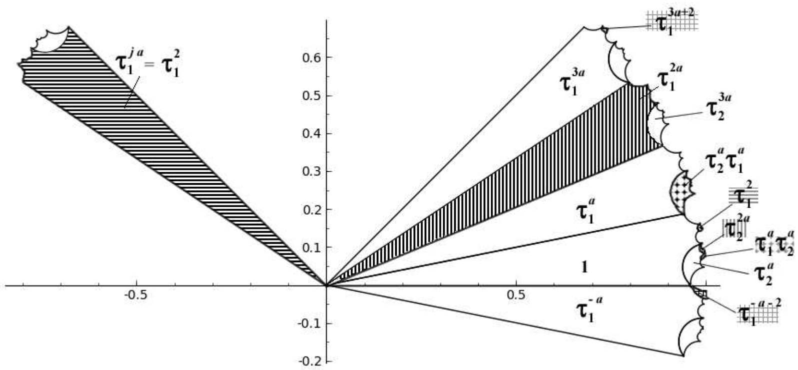
4.1. **Geometric presentation.** We have showed that, for each $S_1 \in \mathfrak{S}_1$, the group

$$G_1 \simeq \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

acts on S_1 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For this action, we have that the 4–tuple

$$(\tau_1^a, \tau_2^a, \tau_1^2, \tau_1^{2^{n-1}c-1}\tau_2)$$

is a generating vector. Using the polygon method [1], according to signature, we find a presentation for G_1. We thus obtain the polygon



We call $D_1\,, D_2\,, D_3$ the elements associated respectively to τ_1^a, τ_2^a, τ_1^2. From the picture we conclude the relationships

$$\begin{array}{rcll} D_3D_1^4D_3 & = & 1, & \text{grid} \\ D_2D_1D_2^{-1}D_1^{-1} & = & 1, & \text{dots} \\ D_1^2D_2^{-2} & = & 1, & \text{vertical lines} \\ D_1^jD_3^{-1} & = & 1, & \text{horizontal lines} \end{array}$$

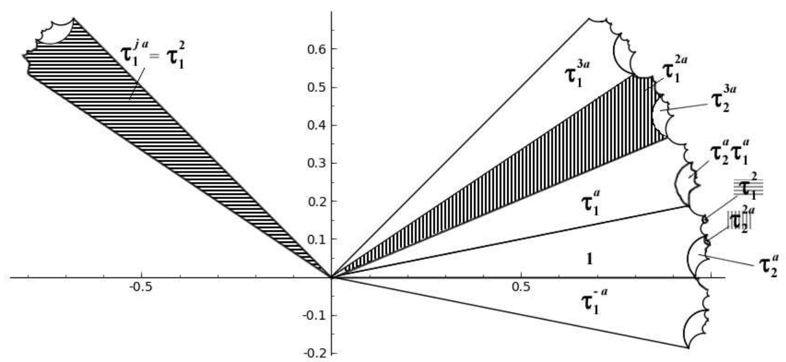
The reader should keep in mind that we are in the case where c is an even number and $2^{n-1}c = 2a+2$. It is not difficult to verify that $j = 2^{n-1}c - 2$.
The following proposition is a consequence.

Proposition 1. *G_1 has a presentation of the form*
(4.3)
$$\left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{c2^{n-1}-2}D_3^{-1} = 1 \end{array} \right\rangle,$$
where c is an even number.

Proof. Call $\widetilde{G}_1$ to a group with presentation (4.3).
First we may see that $D_1D_2 = D_2D_1$. This is because $D_1^2 = D_2^2$ and $D_3 = D_1^{c2^{n-1}-2}$ then $D_4 = D_1^{c2^{n-1}-1}D_2$ and it has order 2.
Now we have
$$\widetilde{G}_1 = \{D_1^jD_2^i : 0 \le j < 2^{n+1}, 0 \le i \le 1\}$$
Then $\widetilde{G}_1$ is an abelian group of order 2^{n+2}. Furthermore $\widetilde{G}_1$ is isomorphic to G_1. This is because we may define an isomorphism Φ given by $\Phi(D_1) = \tau_1$ and $\Phi(D_2) = \tau_2$. □

Also we have showed that, for each $S_2 \in \mathfrak{S}_2$, the group
$$G_2 \simeq \mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$$
acts on S_2 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For this action, we have that the 4–tuple
$$(\tau_1^a, \tau_2^a, \tau_1^2, \tau_1^{2^{n-1}c-1}\tau_2)$$

is a generating vector. As before, using the polygon method [1], according to signature, we find a presentation for G_2. We thus obtain the polygon



We call $D_1\,, D_2\,, D_3$ the elements associated respectively to τ_1^a, τ_2^a, τ_1^2. From the picture we conclude the relationships

$$\begin{aligned} D_1^2 D_2^{-2} &= 1, && \text{vertical lines} \\ D_1^j D_3^{-1} &= 1, && \text{horizontal lines} \end{aligned}$$

The reader should keep in mind that we are in the case where c is an odd number and $2^{n-1}c = 2a+2$. It is not difficult to verify that $j = 3\cdot 2^{n-1}c - 2$, for $n > 3$. For $n = 3$ we have $a = 1$ hence $j = 2$.
The following proposition is a consequence.

Proposition 2. *For $n > 3$, G_2 has a presentation of the form*
(4.4)
$$\left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{3\cdot 2^{n-1}c-2}D_3^{-1} = 1 \end{array} \right\rangle .$$

Proof. Call $\widetilde{G}_2$ to a group with presentation (4.4).
First we will see that $D_2D_1 = D_1^{2^n+1}D_2$. In fact as D_4 has order 2 we have $D_4 = D_1D_2D_3$ then

$$D_2D_1 = D_1^{3-3\cdot 2^n c}D_2^{-1} = D_1^{1-3\cdot 2^n c}D_2$$

but as c is an odd number we have $1 - 3\cdot 2^n c \equiv 2^n + 1 \mod 2^{n+1}$.
Therefore we may define an isomorphism $\Phi : \widetilde{G}_2 \to G_2$ given by $\Phi(D_1) = \tau_1$ and $\Phi(D_2) = \tau_2$. □

4.2. **Classification of actions on $\mathfrak{S}_1$ and $\mathfrak{S}_2$.** In this section we will study the epimorphisms associated to the actions defined in the previous section. From now on we consider the notations as in theorem 2.3. Recall that in any case (abelian and no abelian) $H_1 =< \tau_1 >$ is a subgroup of $\mathrm{Aut}(S_i)$ for $i = 1, 2$, and acts with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$. Then there exist a Fuchsian group $\Gamma_{1,i}$ with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$ and an epimorphism $\theta_{1,i} : \Gamma_{1,i} \to H_1$. We consider for $\Gamma_{1,i}$ a presentation according to your hyperbolic polygon associated

$$\Gamma_{1,i} = \left\langle x_{1,i}, \cdots, x_{5,i} : x_{1,i}^{2^{n+1}} = x_{2,i}^{2^{n+1}} = x_{3,i}^{2^n} = x_{4,i}^{2^n} = x_{5,i}^{2^n} = 1 = x_{1,i}x_{2,i}x_{3,i}x_{4,i}x_{5,i} \right\rangle$$

where for each $j = 1, 2, 3, 4$ we have $x_{j,i}$ is a positive minimal rotation.

Moreover there exists an isomorphisms $f_{1,i} : \Delta/K_{1,i} \to S_i$, where $K_{1,i} = \ker(\theta_{1,i})$. Call $P_{j,i} = f_{1,i}\left(\pi_{K_{1,i}}(z_{j,i})\right)$ then we have

$$\delta_{P_{1,i}}(\theta_{1,i}(x_{1,i})) = \omega_{2^{n+1}}, \quad \delta_{P_{2,i}}(\theta_{1,i}(x_{2,i})) = \omega_{2^{n+1}},$$
$$\delta_{P_{3,i}}(\theta_{1,i}(x_{3,i})) = \omega_{2^n}, \quad \delta_{P_{4,i}}(\theta_{1,i}(x_{4,i})) = \omega_{2^n}$$

By the theorem 3.4 and the tables with the computes of δ_P (case (1), case (4) , case (5)) we have that if

$$P_{1,i} = [0,0,1], \quad P_{2,i} = \tau_2(P_{1,i}), \quad P_{3,i} = [1,0,1], \quad P_{4,i} = [\lambda,0,1],$$

then

$$(\theta_{1,i}(x_{1,i}), \theta_{1,i}(x_{2,i}), \theta_{1,i}(x_{3,i}), \theta_{1,i}(x_{4,i}), \theta_{1,i}(x_{5,i})) = \left(\tau_1^a, \tau_1^{-3a-4}, \tau_1^{2a}, \tau_1^2, \tau_1^2\right)$$

We may do the same process for $H_2 =< \tau_2 >$. It follow there exist a Fuchsian group $\Gamma_{2,i}$ with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$ and an epimorphism $\theta_{2,i} : \Gamma_{1,i} \to H_2$. The group $\Gamma_{2,i}$ has a presentation

$$\Gamma_{2,i} = \left\langle y_{1,i}, \cdots, y_{5,i} : y_{1,i}^{2^{n+1}} = y_{2,i}^{2^{n+1}} = y_{3,i}^{2^n} = y_{4,i}^{2^n} = y_{5,i}^{2^n} = 1 = y_{1,i}y_{2,i}y_{3,i}y_{4,i}y_{5,i} \right\rangle$$

where for each $j = 1,2,3,4$ we have $y_{j,i}$ is a positive minimal rotation.
Moreover there exists an isomorphisms $f_{2,i} : \Delta/K_{2,i} \to S_i$, where $K_{2,i} = \ker(\theta_{2,i})$. Call $Q_{j,i} = f_{2,i}\left(\pi_{K_{2,i}}(z_{j,i})\right)$. If

$$Q_{1,i} = [1,0,1], \quad Q_{2,i} = [-1,0,1], \quad Q_{3,i} = [0,0,1], \quad Q_{4,i} = [\lambda,0,1].$$

then

(1) For c an even number
$$(\theta_{2,1}(y_{1,1}), \theta_{2,1}(y_{2,1}), \theta_{2,1}(y_{3,1}), \theta_{2,1}(y_{4,1}), \theta_{2,1}(y_{5,1})) = \left(\tau_2^a, \tau_2^a, \tau_2^{2a}, \tau_2^2, \tau_2^2\right)$$
(2) For c an odd number
$$(\theta_{2,2}(y_{1,2}), \theta_{2,2}(y_{2,2}), \theta_{2,2}(y_{3,2}), \theta_{2,2}(y_{4,2}), \theta_{2,2}(y_{5,2})) = \left(\tau_2^a, \tau_2^{a(2^n+1)}, \tau_2^{2a}, \tau_2^2, \tau_2^2\right)$$

Theorem 4.5. *Assume that* $2^n - 3a - 5 \neq 0$ *and* $a > 1$ *(case (4) and case (5) theorem 4.1). Then the actions induced by* H_1 *and* H_2 *on* $S_i \in \mathfrak{S}_i$ *for each* $i = 1,2$, *are directly topologically, but not conformally, equivalent, except for* S_i *defined by* $\lambda = \pm 1 \pm \sqrt{2}$.

Proof. We consider the isomorphism $\Phi : H_j \to \mathbb{Z}/2^{n+1}\mathbb{Z}$ and given by $\Phi(\tau_j) = 1$. With this isomorphism we may associate to each generating vector a 5-tuple of elements in $\mathbb{Z}/2^{n+1}\mathbb{Z}$. Thus we have

- If c is an even number, then
$$(a, a, 2a, 2, 2) \equiv (a, -3a-4, 2a, 2, 2) \mod 2^{n+1}.$$
- If c is an odd number, then
$$(a, a(2^n+1), 2a, 2, 2) \equiv (a, -3a-4, 2a, 2, 2) \mod 2^{n+1}.$$

By theorem 3.2 we have in these cases $s = 1$, then the actions are directly topologically equivalent.
Now we will prove that for S_i defined by $\lambda \neq \pm 1 \pm \sqrt{2}$ the actions on S_i are not conformally equivalent.
We will follow the idea of the proof given by G.González-Diez and R.Hidalgo [8] in the case $n = 2$ with $a = 1$ and $c = 2$.

By contradiction we suppose that are conformally equivalent; that is, there exists σ on $\mathrm{Aut}(S_i)$ such that $\sigma\tau_1 = \tau_2^j\sigma$, where j is an odd number.
Now we consider the holomorphic branched covering associated to the action of $H_3 =< \tau_1^2 >$ on S_i, this is $\pi : S_i \longrightarrow \mathbb{C}$ given by $\pi(x, y) = x$.
Hence we have the following diagram

$$\begin{array}{ccc} S_i & \xrightarrow{\ \sigma\ } & S_i \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S_i/H_3 = \widehat{\mathbb{C}} & \overset{T}{\cdots\cdots\to} & \widehat{\mathbb{C}} = S_i/H_3 \end{array}$$

where T is given by $T(x) := \pi(\sigma(x, y))$, for $x \in \widehat{\mathbb{C}}$, where $(x, y) \in \pi^{-1}(x)$.
It is not difficult prove that T is a Möbius transformation.
Consider the set B of the fixed points of H_3 ((4.1)).
Note that $\pi(B) = \{0, \infty, \pm 1, \pm\lambda, \pm\lambda^{-1}\}$ and $\sigma(B) = B$ then $T(\pi(B)) = \pi(B)$. Furthermore $T(0), T(\infty) \in \{1, -1\}$. This is because σ map the fixed points of τ_1 on the fixed points of τ_2.
Now consider the coverings associated to the subgroups H_1 and H_2. Then we have the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\ T\ } & \widehat{\mathbb{C}} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ S_i/H_1 = \widehat{\mathbb{C}} & \overset{R}{\cdots\cdots\to} & \widehat{\mathbb{C}} = S_i/H_2 \end{array}$$

where $\pi_1(x) = x^2$, $\pi_2(x) = x + \frac{1}{x}$ and $R(x) = \pi_2 T(x_0)$, with $\pi_1(x_0) = x$.
For $T_j : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ $(j = 1, 2)$ defined by $T_1(x) = -x$ and $T_2(x) = \frac{1}{x}$ it has that $T_j\pi = \pi\tau_j$ $(j = 1, 2)$. Remark that for each j, π_j is the covering associated to the action T_j. Using these properties we may prove R is a Möbius transformation.
Recall $T(\pi(B)) = \pi(B) = \{0, \infty, 1, \lambda^2, \lambda^{-2}\}$ and $T(\{0, \infty\}) = \{1, -1\}$. Then we have

$$R\left(\{0, \infty, 1, \lambda^2, \lambda^{-2}\}\right) = \left\{\infty, \pm 2, \pm\left(\lambda + \lambda^{-1}\right)\right\}, \tag{4.5}$$

and $R(0), R(\infty) \in \{2, -2\}$.
In the case $R(0) = 2$, $R(\infty) = -2$ and $R(1) = \infty$, we have $R(z) = \dfrac{2z + 2}{-z + 1}$. In particular, $R(\lambda^2) = \dfrac{2(\lambda^2 + 1)}{1 - \lambda^2}$, which must be (by (4.5)) equal to $\pm(\lambda + \lambda^{-1})$. From this, and the fact that $\lambda^2 \neq 1$, we deduce that $\lambda = \pm 1 \pm \sqrt{2}$. In the other case we have the same values as before for λ.
It follow that the action induced by H_1 and H_2 are not conformally equivalent for $\lambda \neq \pm 1 \pm \sqrt{2}$.

□

Remark in the case (1), theorem 4.1 ($n = 3$, $a = 1$ and $c = 1$), it follow the consequences of the preceding theorem. In this case the generating vectors are

$$H_1 : (\tau_1, \tau_1^9, \tau_1^2, \tau_1^2, \tau_1^2), \quad H_2 : (\tau_2, \tau_2^9, \tau_2^2, \tau_2^2, \tau_2^2)$$

and the proof that the actions are not conformally equivalent is the same as that for the preceding theorem.

If $\lambda = \lambda_0 = 1 + \sqrt{2}$, then the automorphism σ is given by

$$\sigma(x, y) = \left(\frac{1-x}{1+x}, \frac{\sqrt{2^c}\, \omega_{2^{n+1}} y}{(x+1)^c} \right)$$

In both cases (c is an even number or c is an odd number) the group generated by τ_1 , τ_2 and σ acts on the Riemann surfaces defined by λ_0 with signature $(0; 2^{n+1}, 2^{n+1}, 4)$.

References

[1] A.Behn, R.Rodríguez and A.Rojas, *Hyperbolic Polygons and Symplectic Representations for group actions on Riemann Surfaces.*
[2] Breuer, *Character and Automorphism Group of Compact Riemann Surfaces*, Cambridge University Press, London Mathematical Society Lecture Note Series, **280**.
[3] Broughton, S.A., *Classifying finite group actions on surfaces of low genus*, J. Pure and Aplied Algebra, Volume **69**, 1990, 233-270.
[4] Farkas and Kra, *Riemann Surfaces*, Springer-Verlag, Volume **71**.
[5] G. González-Diez, *Loci of curves which are prime Galois coverings of P1*, Proc. London Math. Soc., (3), **62** (1991), pp. 469-489.
[6] G. González-Diez, *On prime Galois coverings of the Riemann sphere*, Annali di Matematica pura ed applicata (IV), Vol. CLXVII (1994).
[7] G.González–Diez and W.Harvey, *Moduli of Riemann surfaces with symmetry*, Dicrete groups and geometry, London Math. Soc. Lecture Note Ser. **173**, 1992, 75-94.
[8] G. González–Diez and R.Hidalgo, *Conformal versus topological conjugacy of automorphisms on compact Riemann Surfaces*, Bull. London Math. Soc., **29**, 1997, 280-284.
[9] J. Gilman, *On conjugacy classes in the Teichmuller modular group*, Michigan Math. J., **23**, 1976, 53-63.
[10] Harvey, *On branch Loci in Teichmuller space*, Transactions of the American Mathematical Society, Volume **153**, 1971, 387-399.
[11] S.Katok, *Fuchsian Groups*,The University of Chicago Press, 1992.
[12] L.Keen, *Canonical Polygons for Finitely Generated Fuchsian Groups*, Acta Mathematica 115 (1966), 1-16.
[13] I. Kuribayashi, *On automorphism groups of a curve as linear groups*, J.Math. Soc. Japan, **39**, 1987, no.2, 51-77.
[14] A.M. Macbeath, *The Classification of Non-Euclidean plane Cristallographic Group* , Canad.J.Math., **19**, (1966), 1192-1205.
[15] B. Maskit, *Kleinian groups* , Grundlehren der mathematischen Wissenchaften, v. 287, 1935.
[16] R.Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics, Volume **5**, A.M.S, 1995.
[17] D.Singerman, *Finitely Maximal Fuchsian groups*, J. London Math. Soc., **6**, 1972, 29-38.

Universidad de Chile
E-mail address: marie.carvacho@gmail.com