Nº PROYECTO : 3085013  
DURACIÓN : 2 años  
AÑO ETAPA : 2009  

TÍTULO PROYECTO : NON-LINEAR SYMMETRIES AND SUPERSYMMETRY.  

DISCIPLINA PRINCIPAL : MECANICA QUÁNTICA  
GRUPO DE ESTUDIO : FISICA TEORICA Y EXP  
INVESTIGADOR(A) RESPONSABLE : VIT JAKUBSKY  
DIRECCIÓN :  
COMUNA :  
CIUDAD :  
REGIÓN : METROPOLITANA
INFORME FINAL  
PROYECTO FONDECYT POSTDOCTORADO  

**OBJETIVOS**  
Cumplimiento de los Objetivos planteados en la etapa final, o pendientes de cumplir. Recuerde que en esta sección debe referirse a objetivos desarrollados, NO listar actividades desarrolladas.

<table>
<thead>
<tr>
<th>Nº</th>
<th>OBJETIVOS</th>
<th>CUMPLIMIENTO</th>
<th>FUNDAMENTO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Investigation of hidden nonlinear symmetries in (quasi) exactly solvable systems - exploring the structure of nonlinear supersymmetry within the system described by finite-gap associated Lame equation</td>
<td>TOTAL</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Study of nonlinear (hidden) supersymmetry in general class of finite-gap systems with real smooth and parity even potentials</td>
<td>TOTAL</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Exploring higher-order supersymmetry with use of advanced quantization techniques in Aharonov-Bohm effect on AdS_2 space</td>
<td>TOTAL</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Exploration of hidden supersymmetry in planar Aharonov-Bohm problem</td>
<td>TOTAL</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Study of systems with anomaly-free supersymmetry and additional anti-linear symmetries</td>
<td>TOTAL</td>
<td></td>
</tr>
</tbody>
</table>

Otro(s) aspecto(s) que Ud. considere importante(s) en la evaluación del cumplimiento de objetivos planteados en la propuesta original o en las modificaciones autorizadas por los Consejos.
RESULTS OBTAINED:
For each specific goal, describe or summarize the results obtained. Relate each one to work already published and/or manuscripts submitted. In the Annex section include additional information deemed pertinent and relevant to the evaluation process.

The maximum length for this section is 5 pages. (Arial or Verdana, font size 10).

The main subject of the proposed project was the investigation of physical systems with non-linear symmetries. We have studied the systems in both classical and quantum framework. The results obtained during the evaluation of the project support our initial working hypothesis, stated in the proposal: particular properties of physical systems are reflected by presence of possibly nonlinear symmetries or supersymmetries in the system.

In the following text, we present the main research directions that we followed. We briefly review the results that have been obtained and link them to the published/submitted papers.

- **Investigation of hidden nonlinear symmetries in (quasi) exactly solvable systems**
  
  *Exploring the structure of nonlinear supersymmetry within the periodic finite-gap systems with even smooth potential*

  We focused to the periodic quantum systems with finite number of spectral gaps [1], [2]. These systems play an important role in physics: they underlie the theory of periodic solutions in nonlinear integrable models, including the systems based on Korteweg-de Vries, nonlinear Schrödinger or sine-Gordon equations. Besides, they find their application in condensed matter theory, in the theory of monopoles, Josephson junction theories, chaos or classical Ginzburg-Landau field theory, see [2] for references.

  The spectral properties of periodic finite-gap systems suggest on the presence of a hidden supersymmetry; the energy levels in the interior of the bands are doubly-degenerated while the band-edge states are singlets. We initiated our study with the quantum systems described by the finite-gap associated Lamé equation [1]. Its self-isospectral extension can be considered as a planar model of a non-relativistic electron in the external electromagnetic field. We revealed three nontrivial integrals of motion that form a rich algebraic structure together with three trivial symmetries of the system. Any of the trivial symmetries can be selected as a $Z_2$-grading of the superalgebra. The qualitative structure of the superalgebra is independent on the actual choice of the grading operator. We named this structure tri-supersymmetry and demonstrated that it faithfully reflects the spectral characteristics of the system.

  In [2], we generalized these results to a broad class of the finite-gap periodic systems with smooth and parity-even potentials. A finite-gap potential of a periodic quantum system is a solution of one (and hence infinitely many) higher-order nonlinear equation of the stationary KdV hierarchy. The supercharge of the hidden bosonised supersymmetry together with the Hamiltonian form the Lax pair - the key object in the theory of integrable systems. Different factorizations of the odd-order Lax operator give rise to a rich algebraic structure consisting of several nonlinear integrals of motion. We posed a conjecture of self-isospectrality, which tells how to construct self-isospectral system with use of higher-order Crum-Darboux transformation. We elaborated the associated Lamé system as a detailed example of these general results and considered its infinite-period limit which coincides with the super-extended reflectionless Pöschl-Teller system. We provided an explicit form for its higher-dimensional supercharges.
Exploring higher-order supersymmetry of Aharonov-Bohm system in $AdS_2$ space with use of advanced quantization techniques

Our main objective was to acquire an insight into the origin of the tri-supersymmetric structure of the reflectionless Pöschl-Teller (P-T) model that we observed in [2]. We considered a nonrelativistic particle moving on $AdS_2$ space in presence of a singular magnetic flux. The Hamiltonian of the system was proportional to the Casimir operator of Lie algebra $so(2,1)$. We discussed several ways of quantization of the two-dimensional Hamiltonian.

It was the essential point of the analysis that the subsystems corresponding to fixed half-integer values of shifted angular momentum correspond to the reflectionless P-T systems. Insensitivity of the Hamiltonian with respect to the sign of angular momentum made it possible that the identical P-T systems appeared in two different subspaces. We found the relation of the nonlinear integrals of motion of P-T system and the ladder operators of $so(2,1)$. The intertwining relations of P-T model were rewritten as matrix elements of the vanishing commutator between Hamiltonian and monomial of the ladder operators. We discussed the role of discrete symmetries of the two dimensional system in the context of tri-supersymmetry of P-T system as well. We employed the relation of the reflectionless P-T system with the free particle and constructed a deformed conformal symmetry of the Pöschl-Teller system.

Exploration of hidden supersymmetry in planar Aharonov-Bohm problem

We analyzed a planar quantum system of a spin-less particle moving in the field of a singular impenetrable magnetic flux. The flux punctured the plane perpendicularly in the origin. Qualitatively the same model has been proposed originally in the seminal work of Aharonov and Bohm, where they demonstrated the fundamental role of electromagnetic potential in the quantum physics.

It is a well known fact that the system described above is not defined uniquely until a self-adjoint extension of the Hamiltonian is fixed. The different self-adjoint extensions can be attributed to different quantum mechanical realizations of the impenetrability of the magnetic flux or internal properties of the flux. We tested the model on presence of a hidden supersymmetry.

We constructed a non-local supercharge that formally satisfies all the required properties. Then we applied the theory of self-adjoint extensions of symmetric operators to find the admissible domain of the supercharge. The Hamiltonian, defined as a square of the supercharge, is then self-adjoint by construction. We found that there are at least three self-adjoint extensions that support existence hidden supersymmetry. One of them coincides with the original setting studied by Aharonov and Bohm, the other two models allow singular behavior of wave functions near the origin. During the analysis, we observed a remarkable correlation between

existence of the hidden supersymmetry and scale-invariance of the systems. We find it as a very interesting interplay between the two physically distinct types of symmetries.

The conformal invariance of these systems was the initial point in construction of superconformal symmetry of the model. New nontrivial dynamical symmetries of the system were found. We provided an alternative interpretation of the model in terms of interacting anyons and discussed briefly how is the supersymmetry reflected in the calculation of the scattering amplitude.


• Study of systems with anomaly-free supersymmetry and additional anti-linear symmetries

In [5], we considered a one-dimensional quantum system in presence of a potential which coincides asymptotically with $V(x) \sim -x^4$. To maintain the square-integrability of solutions of the stationary Schrödinger equation, the coordinate has been shifted into the complex plane in the way that it spreads over several Riemann sheets. This procedure breaks hermiticity of the Hamiltonian. However, it possesses an antilinear symmetry, that can be written as a product of parity $P$ and time-reversal operator $T$. The spectrum of the system can be acquired in straightforward way due to the analyticity of the potential. The nontrivial topology of the complex plane makes it necessary to discuss various possible definitions of the operator $T$.


As we stated in the beginning, our working hypothesis has been verified in various systems that we discussed above. We observed that the finite-gap periodic quantum systems with parity even potential have a hidden supersymmetry. We found an alternative explication of the hidden supersymmetry within reflectionless Pöschl-Teller system. The scale-invariance of the spin-less systems with singular magnetic flux is in correlation with the hidden supersymmetry given in terms of nonlocal integrals of motion.

We feel encouraged by these results. We intent to continue the study of the nonlinear symmetries and to extend our insight into the relation of the intrinsic properties of physical systems and the associated nonlinear algebraic structures.
ACHIEVEMENTS OF THE PROJECT:
- Research visit(s) to other institution(s).
- Outreach activities related to the projects main topic.
- Any other contribution, not addressed elsewhere, that you consider important.

The maximum length for this section is 1 page. (Arial or Verdana, font size 10).

- research visit, CECS Valdivia, Chile, October 26-27, 2009,
  - talk given: “The hidden nonlinear supersymmetry of Pöschl-Teller model via geometrical symmetries of $AdS_2$ space”, October 27, 2009

- research visit, Universite de Tours, France, June 17-19, 2009,

- participation in the international conference “Representation theory in Mathematics and in Physics” June 11-13, Strasbourg, France
PROJECT SUMMARY:
Using non-specialist language, provide a precise and brief description of the project goals, objectives and results obtained. This summary may be published in CONICYT’s web page.

The maximum length for this section is 1 page. (Arial or Verdana, font size 10).

The main subject of the proposed project was the investigation of non-linear symmetries within physical systems. We assumed that there is a correlation between intrinsic properties of particular systems and existence of their hidden non-linear symmetries. We tested this hypothesis on various physically interesting settings.

We analyzed periodic quantum systems with finite number of spectral gaps. These systems serve as an approximation of generic smooth periodic potentials. They found application in various areas of physics. Namely, they are particularly useful in modeling of crystals, they appear in the theory of monopoles. Besides, the finite-gap potentials are intimately related to the theory of periodic solutions in nonlinear integrable models (KdV stationary hierarchy).

We found that the spectral band structure is intimately related to the rich superalgebraic structure, composed of nontrivial integrals of motion. The integrals reflect the spectrum of the system, its band structure and structure of band-edge states.

As an example, we elaborated the explicit form of the discovered superalgebraic structure for the quantum system governed by associated Lamé equation. The infinite-period limit of the system coincides with the reflectionless Pöschl-Teller system and inherits the supersymmetric structure of the associated Lamé system. We explained the nature and origin of the superalgebra with use of the geometrical symmetries of a higher-dimensional system living in $AdS_2$ space. We found that appearance of the hidden supersymmetry is conditioned by reflectionlessness of Pöschl-Teller potential.

We analyzed a two-dimensional quantum system, represented by a spin-less particle in presence of an impenetrable singular magnetic flux. Historically, the model was used for a theoretical study of Aharonov-Bohm effect, an experimentally observed phenomenon that manifests quantum character of the nature. We found that the system possesses a hidden supersymmetry composed by nonlocal integrals of motion. We revealed the same algebraic structure in two similar systems where the impenetrability of the magnetic flux is realized by different boundary conditions. During the analysis, we observed a remarkable correlation between existence of the hidden supersymmetry and scale-invariance of the Hamiltonians, which we find as an unexpected interplay between two completely distinct types of physical symmetries.
The Project of Vit Jakubský was devoted to investigation of diverse aspects of hidden nonlinear symmetries and supersymmetries. Various important results were obtained on the subject. These include:

- the discovery of a new, rich nonlinear supersymmetric structure in a certain generalization of the Landau problem, that was called a "tri-supersymmetry";

- on its basis, a phenomenon of self-isospectrality, observed earlier by G. Dunne and J. Feinberg in some particular finite-gap quantum periodic systems, was explained and generalized for a general case of finite-gap periodic systems;

- infinite period limit of the tri-supersymmetry was investigated and clarified;

- hidden nonlinear supersymmetry of a reflectionless Poschl-Teller system was explained in the light of AdS/CFT holography and Aharonov-Bohm effect;

- hidden superconformal symmetry was revealed in the Aharonov-Bohm effect for a spinless particle, and was explained in the light of translation invariance breaking.

The results of the research have been presented in 4 journal publications, including a paper published in Physical Review Letters, and in one paper submitted for publication. They were also presented on several International and Chilean Workshops, as well as on various seminars given in European and Chilean Universities. According to the SLAC data base, these papers have accumulated already 44 citations.

I evaluate the work done by Vit Jakubský to be classified within the highest-rank category.
## ARTÍCULOS

Para trabajos en Prensa/ Aceptados/Enviados adjunte copia de carta de aceptación o de recepción.

### Artículo 1

<table>
<thead>
<tr>
<th>Ítem</th>
<th>Detalle</th>
</tr>
</thead>
<tbody>
<tr>
<td>N°</td>
<td>1</td>
</tr>
<tr>
<td>Autor (a)(es/as)</td>
<td>Correa, F; Falomir, H; Jakubsky, V; Plyushchay, M.</td>
</tr>
<tr>
<td>Nombre Completo de la Revista</td>
<td>Journal of Physics A: Mathematical and Theoretical</td>
</tr>
<tr>
<td>Título (Idioma original)</td>
<td>Hidden superconformal symmetry of spinless Aharonov-Bohm system</td>
</tr>
<tr>
<td>Indexación</td>
<td>ISI</td>
</tr>
</tbody>
</table>

Estado de la publicación a la fecha: Enviada

Otras Fuentes de financiamiento, si las hay:

Envía documento en papel: Sí
Archivo(s) Asociado(s) al artículo:
- AB.pdf

---

### Artículo 2

<table>
<thead>
<tr>
<th>Ítem</th>
<th>Detalle</th>
</tr>
</thead>
<tbody>
<tr>
<td>N°</td>
<td>2</td>
</tr>
<tr>
<td>Autor (a)(es/as)</td>
<td>Znojil, M; Jakubsky, V.</td>
</tr>
<tr>
<td>Nombre Completo de la Revista</td>
<td>Pramana - Journal of Physics</td>
</tr>
<tr>
<td>Título (Idioma original)</td>
<td>Supersymmetric quantum mechanics living on topologically nontrivial Riemann surfaces</td>
</tr>
<tr>
<td>Indexación</td>
<td>ISI</td>
</tr>
</tbody>
</table>

Año: 2009
Vol.: 73
N°: 2
Páginas: 397-404

Estado de la publicación a la fecha: Publicada

Otras Fuentes de financiamiento, si las hay:

Envía documento en papel: Sí
Archivo(s) Asociado(s) al artículo:
- Pramana.pdf
<table>
<thead>
<tr>
<th>N°</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Autor (a)(es/as) :</td>
<td>Correa, F; Jakubsky, V; Plyushchay, M.</td>
</tr>
<tr>
<td>Nombre Completo de la Revista :</td>
<td>Journal of Physics A: Mathematical and Theoretical</td>
</tr>
<tr>
<td>Título (Idioma original) :</td>
<td>Finite-gap systems, tri-supersymmetry and self-isospectrality</td>
</tr>
<tr>
<td>Indexación :</td>
<td>ISI</td>
</tr>
<tr>
<td>ISSN :</td>
<td></td>
</tr>
<tr>
<td>Año :</td>
<td>2008</td>
</tr>
<tr>
<td>Vol. :</td>
<td>41</td>
</tr>
<tr>
<td>N° :</td>
<td>48</td>
</tr>
<tr>
<td>Páginas :</td>
<td>485303</td>
</tr>
<tr>
<td>Estado de la publicación a la fecha :</td>
<td>Publicada</td>
</tr>
<tr>
<td>Otras Fuentes de financiamiento, si las hay :</td>
<td></td>
</tr>
</tbody>
</table>

Para descargar el documento, visite [este enlace](http://sial.fondecyt.cl/index.php/investigador/f4_articulos/descarga/22365071/3085013/2009/4375/1/).
OBRAS PUBLICADAS / PRODUCTOS

Syn informacion ingresada.

CONGRESOS

N° : 1
Autor (a)(es/as) : Vit Jakubsky
Título (Idioma original) : Non-linear supersymmetry of Posch-Teller model from the perspective of AdS_2 space
Nombre del Congreso : Simetrías fundamentales y más allá
País : CHILE
Ciudad : Afunalhue
Fecha Inicio : 11/01/2009
Fecha Término : 13/01/2009
Nombre Publicación :
Año :
Vol. :
N° :
Páginas :
Envía documento en papel : si
Archivo asociado : Afunalhue.pdf

N° : 2
Autor (a)(es/as) : Vit Jakubsky
<table>
<thead>
<tr>
<th>Nº</th>
<th>Autor (a)(es/as)</th>
<th>Título (Idioma original)</th>
<th>Nombre del Congreso</th>
<th>País</th>
<th>Ciudad</th>
<th>Fecha Inicio</th>
<th>Fecha Término</th>
<th>Páginas</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Vit Jakubsky</td>
<td>Associated Lame system and its isospectral superpartners</td>
<td>Quantum Physics with Non-Hermitian Operators</td>
<td>ESPANA</td>
<td>Benasque</td>
<td>29/06/2008</td>
<td>11/07/2008</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>Vit Jakubsky</td>
<td>Isospectral super-extensions of the associated Lame system</td>
<td>The XVII International Colloquium on Integrable Systems and Quantum symmetries</td>
<td>REPUBLICA CHECA</td>
<td>Prague</td>
<td>19/06/2008</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


Fecha Término : 21/06/2008
Nombre Publicación :
Año :
Vol. :
Nº :
Páginas :
Envía documento en papel : no
Archivo Asociado :
Praha.pdf

ANEXOS
A continuación se detallan los anexos físicos/papel que no se incluyen en el informe en formato PDF.

1. The talk given in CECS Valdivia, Chile, October 28, 2009
2. The talk given in the international conference "Symmetries in nonlinear mathematical physics", Kiev, Ukraine, June 26, 2009
3. The talk given in Universite de Tours, France, June 18, 2009
4. Submission confirmation from Journal of Physics A
5. The paper submitted to Journal of Physics A
6. The article published in Pramana - Journal of Physics
7. The article published in Annals of Physics
8. The talk given in the conference "Simetrias fundamentales y mas alla"