GEOMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

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I would like to dedicate this work to my wife Valeria and to my son Bruno, who have been giving me love and dedication every single day, since I started graduate school. This work would not have been this rewarding without their support.
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Andres Zuniga

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We study existence questions and qualitative properties of solutions to variational problems related to minimization of geometric quantities, such as generalized notions of length of curves and of the area of surfaces, in a suitable sense. In the first part we consider a nonnegative multi-well potential function and we study the existence of heteroclinic orbits joining distinct minima of the potential, which are defined as classical solutions to a system of second order ordinary differential equations of gradient type. We find a geometric condition on the wells that guarantees the existence of heteroclinic connection between the wells. When this condition is met, the heteroclinic is built as a re-parametrization of a geodesic curve that minimizes a distance related to the potential, between the two wells. Next, we prove these heteroclinic orbits can be characterized as limits, in an appropriate sense, of sequences of periodic solutions to the system of ordinary differential equations previously considered, where each orbit joins the two connected components of the same level set of the potential, as the value of the level set approaches the global minimum.

In the second part we prove the existence and show regularity of functions that minimize an inhomogeneous version of the total variation functional on a fixed domain and subject to Dirichlet data, in arbitrary dimensions. Assuming, among other things, that the weight function is positive and the continuity of the Dirichlet data, we adapt the procedure in [80] by constructing a continuous solution of this problem level set by level set, where each level set is a hypersurface minimizing an area related to the weight function, amongst competitors compatible with the boundary data.
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<td>$H(U)$</td>
<td>Lagrangian energy of the curve $U$</td>
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<td>$\ell_W(U)$</td>
<td>length of the curve $U$ induced by the potential $W$</td>
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<td>$d_W(p_j, p_k)$</td>
<td>distance between points $p_j$ and $p_k$ induced by the potential $W$</td>
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<td>Two components of $c$-sublevel set of $W$</td>
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<td>$H^c(q)$</td>
<td>Lagrangian energy of $q$ renormalized at level $c$</td>
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<td>$\mathcal{M}_c$</td>
<td>admissible class for the variational problem (BTPc)</td>
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<td>$H_{\text{aff}}^1(\mathbb{R}; \mathbb{R}^N)$</td>
<td>affine Sobolev space equal to $H^1(\mathbb{R}; \mathbb{R}^N) + U_{\text{aff}}$</td>
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<td>$U_{\text{aff}}$</td>
<td>affine map joining the wells $p_-$ to $p_+$ of $W$</td>
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<td>$[p_1, p_2]$</td>
<td>linear segment in $\mathbb{R}^N$ between $p_1$ and $p_2$</td>
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<td>$\int_U \alpha(x)</td>
<td>Du</td>
<td>$</td>
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<tr>
<td>$\mathcal{P}_\alpha(E, B)$</td>
<td>$\alpha$-perimeter of the set $E$ in $B$</td>
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<tr>
<td>$d\mathcal{H}^{N-1}$</td>
<td>$(N-1)$-dimensional Hausdorff measure in $(\mathbb{R}^N,</td>
<td>\cdot</td>
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<tr>
<td>$BV_g(\Omega)$</td>
<td>BV functions on $\Omega$ with trace equal to $g$ on $\partial\Omega$</td>
<td>20</td>
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<tr>
<td>${u \geq t}$</td>
<td>$t$-superlevel set of $u$ in $\mathbb{R}^N$</td>
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<td>$\emptyset$</td>
<td>zero level set of the potential $W$</td>
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<td>$B_r(p)$</td>
<td>Euclidean ball in $\mathbb{R}^N$ of center $p$ and radius $r$</td>
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<td>$\mathcal{T}_\gamma^W$</td>
<td>$\mathcal{T}_\gamma^W = {t : W(\gamma(t)) = 0}$</td>
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<tr>
<td>$\text{Lip}_W(p, q)$</td>
<td>Lipschitz curves with respect to $W$ joining $p$ to $q$</td>
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<td>$E(\gamma)$</td>
<td>Jacobian energy of the curve $\gamma$ induced by $W$</td>
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$(\tau_{c,\beta}^-, \tau_{c,\beta}^+)$ largest open interval of time containing 0 on which $W(q_c(t)) > \beta$  
 $\mathcal{B}_{c,\beta}$  $\mathcal{B}_{c,\beta} = \{t \in \mathbb{R} : W(q_c(t)) > \beta\}$  
 $U_{c,\beta}$ gradient flow extension of $U_c$ from level $\beta$  
 $q_{c,\beta}$ gradient flow extension of $q_c$ from level $\beta$  
 $T_c^-, T_c^+$ distinguished times where $\Phi_1 \in \{W = c\}^-$ and $\Phi_2 \in \{W = c\}^+$  
 $q_{c,\beta}|_{\{W \geq c\}}$ truncation at level $c$ of the gradient flow extension $q_{c,\beta}$  
 $M_0$ Asymptotic upper bound of $H^0(q_{c,2c})$  
 $\partial M E$ measure-theoretic boundary of $E$  
 $\overline{\Theta(E,x)}$ upper density of $E$ at $x$  
 $\partial^* E$ reduced boundary of $E$  
 $\mu \perp f$ Radon measure $\nu$ so $\nu << \mu$ for which $\frac{d\nu}{d\mu} = f$  
 $\text{reg}(\partial E)$ set of points where $\partial E$ is locally a smooth hypersurface  
 $\text{sing}(\partial E)$ set of singularity points of $\partial E$  
 $\mathcal{M}_\alpha u$ $\alpha$-minimal surface operator of $u$  
 $\mathcal{H}_g^\alpha$ $\alpha$-dimensional Hausdorff measure on a manifold induced by the geodesic distance $d_g$  
 $\mathcal{D}^L(U)$ set of $L$-differential forms in $M$ with support in $U \subset M$  
 $[E]$ $N$-dimensional current induced by a set $E \subset \mathbb{R}^N$  
 $\partial T$ boundary of the current $T$  
 $\mathcal{M}_{U,g}(T)$ mass of the current $T$ in $U$ with respect to metric $g$  
 $\mathcal{L}_t$ $t$-superlevel set of the extension $G$ of boundary data $g$  
 $E_t$ subset in $\mathbb{R}^N$ solving problems $(\star_t)-(\star\star_t)$  
 $A_t$ $A_t := (E_t \cap \Omega)$
Chapter 1

Introduction

The goal of this dissertation is to study three problems in the calculus of variations which share a common feature, namely, the minimization of quantities bearing a geometric nature. In particular, we concentrate on geometric notions such as generalized length of curves, generalized areas of hypersurfaces, and gradient flows. The main topics of this thesis are

(I) Existence of heteroclinic connections of multi-well gradient systems.

(II) Convergence of vector-valued periodic solutions to a heteroclinic connection.

(III) Regularity of functions minimizing a weighted total variation functional.

Our study of (I) is closely related to that of curves minimizing some kind of length, in (II) we will use solutions to gradient flows, whereas we explore (III) by studying a problem for sets related to the theory of generalized minimal surfaces. These geometric objects play a crucial role in obtaining a description of the key features arising in the solution of these three problems.

In the first part of this thesis we address topics (I) and (II), both focusing on different aspects of solutions to the same class of ordinary differential equations. The main objects of study in topic (I) are classical solutions \( U : (-\infty, +\infty) \rightarrow \mathbb{R}^N \) to second order systems of differential equations of gradient type

\[
U''(t) = \nabla_u W(U(t)) \quad \text{for} \quad t \in (-\infty, +\infty),
\]

where \( W = W(u) \) is a non-negative \( C^1 \)-function with a finite number \( m \geq 2 \) of distinct global
minima $p_1, \ldots, p_m \in \mathbb{R}^N$ at value zero, for some dimension $N \geq 2$. When a solution of (1.1) connects any two of the wells $p_j, p_k$ at infinity, namely,
\begin{equation}
U(-\infty) = p_j, \quad U(\infty) = p_k, \quad \text{with } j \neq k,
\end{equation}
such solution is called a heteroclinic connection between $p_j$ and $p_k$.

Physically, problem (1.1)-(1.2) can be interpreted in classical mechanics as the Newtonian Law of motion
\[ \vec{F}(t) = m \vec{a}(t) \]
of a particle with mass $m = 1$, in an environment with force term $\vec{F} = -\nabla(-W)$ due to the potential function $-W$, and with no presence of dissipative (friction) terms. In this context, $U(t)$ represents the trajectory of an ideal unit-mass particle going from a global maximum $p_j$ of the potential to another one $p_k$, asymptotically in time. By allowing a little misuse of language, from now on we will simply refer to $W$ as the potential function.

The considerable interest in heteroclinic connections arises in part from the central role they play in the theory of phase transitions [33,34,49,50] as well as in studies of pattern formation [45]. In this context, given $\varepsilon > 0$ sufficiently small and $N \geq 1$, one is interested in describing non-constant local minimizers $u_\varepsilon : \Omega \to \mathbb{R}^N$ of the variational problem
\begin{equation}
\inf_{u \in H^1(\Omega)} E_\varepsilon(u) := \int_{\Omega} \left\{ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right\} \, dx,
\end{equation}
where $W$ is a non-negative multi-well potential, $\Omega \subset \mathbb{R}^n$ is a bounded domain for some $n \geq 2$. For $\varepsilon > 0$ chosen appropriately small we can anticipate the structure of a non-constant local minimizer of (1.3): it should stay close to one well $p_j$ of $W$ on one portion of $\Omega$ and close to another well $p_k$ ($j \neq k$) on the other part, with a rapid transition layer bridging these two wells. Heteroclinic connections are used precisely to describe the profile of the order parameter $u$ as it varies in the normal direction along the interface. To see this intuitively, let us observe that a heteroclinic solving (1.1)-(1.2) corresponds, up to a scaling factor of $\varepsilon^2$, to a one-dimensional solution to the
Euler-Lagrange equation associated to (1.3)

\[ \varepsilon \Delta u = \frac{1}{\varepsilon} \nabla u W(u) \quad \text{in} \quad \Omega, \quad (1.4) \]

a semilinear elliptic system of partial differential equations, commonly known as the Allen-Cahn system. First, with a point of view of energy analysis, is that the authors in \([11,78,79]\) introduce the use of heteroclinic connections to rigorously address the problem of $\Gamma$-convergence of the family of variational problems (1.3) as $\varepsilon \to 0^+$, in the case where $u$ is vector-valued ($N \geq 2$). Minimizers of $E_\varepsilon$ exhibiting multiple transitions between phases have been constructed later on using heteroclinic connections for some classes of domains $\Omega$, see e.g. \([82]\). Switching now to the point of view of differential equations, years later in \([20]\) the authors utilize multiple heteroclinic connections to construct solutions in the planar setting $n = 2, N = 2$ to (1.4) with triple junctions, i.e. having three transition layers where $u$ is close to the wells, by assuming symmetry properties of the potential (equivariance). Subsequently, a major focus of interest in the literature throughout the years has been building layered solutions to the Allen-Cahn and Cahn-Hilliard systems, see \([3, 5, 28, 53, 69]\), using a variety of different methods, in all of which heteroclinic connections play a fundamental role.

In a different vein, the gradient flow of the energy functional $E_\varepsilon$ as defined in (1.3) induces the parabolic version of the Allen-Cahn system

\[ \partial_t u = \varepsilon \Delta_x u - \frac{1}{\varepsilon} \nabla_x W(u), \quad (1.5) \]

\[ u(t, x) : (0, \infty) \times \Omega \to \mathbb{R}^N, \]

where $\Omega \subset \mathbb{R}^n$, which serves as a model for simulating the evolution of grain boundaries, through the analysis of interface motion and junctions formation. In this setting heteroclinic connections correspond to standing waves to the gradient diffusion system

\[ \partial_t u = \partial^2_y u - \nabla_y W(u), \]

\[ u(t, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}^N. \]
For small $\varepsilon$, the authors in [16,17,21] show that after $O(1)$ amount of time after starting (1.5) with typical initial data, $u$ near the interface and away from the junction is essentially described by a heteroclinic connection. This is plausible since at the interface the Laplacian and the free term balance each other, to leading order in $\varepsilon$, thus (1.1) is satisfied approximately. Other applications include, see [2, 24], the construction of traveling wave solutions in the plane to the Hamiltonian flow system induced by $E_{\varepsilon}$ in (1.3),

$$J \partial_t u = \Delta_x u - \nabla_u W(u) \quad \text{for} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the study of interphase boundaries in a multi-order-parameter phase-field model for the description of crystallography of materials [7].

The first problem of this thesis in Chapter 2, consists of studying the existence, and non-existence, of heteroclinic connections solving (1.1)-(1.2) for a given potential function. Let us observe that even in the scalar case $N = 1$ the existence of a connection between any pair of wells is not guaranteed for potentials with more than two global minima. Indeed, for the potential in the Figure 1.1 below one can easily see using uniqueness of solutions to ODE that an obstruction phenomenon occurs, and it is a global one.

![Figure 1.1: Obstruction of heteroclinic connections between $p_1$ and $p_3$.](image)

Although results in this direction are already available in the literature (to be described below), our main motivation is rooted in giving a simpler and perhaps more elegant proof of existence of such heteroclinic connections in arbitrary dimensions $N \geq 2$. 

4
Existence of vector-valued heteroclinic connections has been established more than two decades ago in [68] and [79], by means of variational methods, utilizing natural assumptions on \(W\) about smoothness and non-degeneracy around the wells:

\[
W \in C^2(\mathbb{R}^N), \quad D^2W(p) > 0 \quad \text{at the wells.}
\]

Over recent years, however, progress has been made in order to weaken the hypothesis on \(W\) and for a variety of hypotheses on the values of \(m\) and \(N\), including [9, 41, 76], where the authors base their analysis on finding critical points or minimizers of the associated Lagrangian energy

\[
H(U) := \int_{-\infty}^{+\infty} \frac{1}{2} |U'|^2 + W(U).
\]

Let us remark that a “naive” strategy to retrieve heteroclinic connections minimizing \(H\) over a natural class of admissible maps is to try to solve the variational problem

\[
\inf \{ H(U) : U \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N), \ U(-\infty) = p_j, \ U(+\infty) = p_k \} \quad (1.6)
\]

by the direct method in the calculus of variations. However, such a strategy fails to establish existence of a minimizer in general, due to the translation invariance of \(H\). This property causes a loss of compactness in reasonable topologies (such as pointwise convergence) for limits to belong in the admissible class. One can come up with a minimizing sequence \(\{U_i\}\) for (1.6) rapidly transitioning between the wells \(p_j, p_k\) on a finite time interval, so if \(t_i \to +\infty\) then \(\tilde{U}_i(t) := U_i(t-t_i)\) is still a minimizing sequence but \(\tilde{U}_i(t) \to p_-\) locally uniformly as \(i \to \infty\). It is for this reason is that in [9, 41, 76] the authors deal with finding critical points either by utilizing a mountain pass method, or a constrained minimization suited to circumvent the translation invariance issue. This issue is relevant not only for this problem in Chapter 2 but will also be crucial for the next problem in Chapter 3 (cf. §3.2.2).

A secondary motivation for this work is to determine a wider class of potentials for which existence of heteroclinic connections connecting two minima of the potential is guaranteed, where
the focus of attention is on the non-degeneracy hypothesis of the potential at the wells and the regularity of \( W \), for arbitrary values of \( m, N \geq 2 \).

Here we return to the approach of [79], originally introduced in [78] for a related problem where the potential vanishes along two planar curves. In the case of a planar system \( N = 2 \) and a double-well potential \( m = 2 \) existence was established in [79] under somewhat stringent non-degeneracy assumptions on the behavior of \( W \) near the wells. Now we place the existence question within the context of minimizing geodesics in length spaces. Under quite weak assumptions on \( W \) near the wells, we provide a simple proof of existence for solutions to (1.1) for \( m = 2 \) and \( N \) arbitrary, as well as a geometric characterization of sufficient conditions for existence that hold for any \( m \geq 3 \) and any \( N \geq 2 \). The realization of heteroclinic connections as minimizing geodesics, in a sense to be described below, yields a more geometric characterization of these curves in phase space than one typically gets from minimization of \( H \).

The starting point for the approach here, as well as in [79], is the observation that heteroclinic connections enjoy the property of equipartition of energy, namely

\[
\int_{-\infty}^{+\infty} \frac{1}{2} |U'|^2 = \int_{-\infty}^{+\infty} W(U).
\]

Consequently, viewing \( H(U) \) as a sum of squares, one sees that heteroclinic connections yield equality in the trivial inequality \( H(U) \geq \sqrt{2} \ell_W(U) \) satisfied by any competitor, where

\[
\ell_W(U) := \int_{-\infty}^{+\infty} \sqrt{W(U)} |U'|.
\]

This naturally leads one to consider the minimization problem

\[
\inf\{\ell_W(U) : U(\pm\infty) = p_j, U(\pm\infty) = p_k\} \quad \text{for} \quad j, k \in \{1, 2, \ldots, m\}. \tag{1.7}
\]

We observe that (1.7) is purely geometric, with the value of \( \ell_W \) depending only on a curve, not on its parametrization, so one regards this as a problem of minimizing the distance between \( p_j \) and \( p_k \) in a degenerate Riemannian metric having a (degenerate) conformal factor \( \sqrt{W} \), a metric denoted
here by \( d_W(p_j, p_k) \). It follows immediately from the use of a parametrization of equipartition of energy, namely, one in which a minimizer of \( \ell_W \) obeys a pointwise equipartition of energy

\[
\frac{1}{2} |U(x)|^2 = W(U(x)) \quad \text{for } x \in \mathbb{R},
\]

that a minimizer of \( \ell_W \) yields a minimizer of \( H \), and hence a solution to the heteroclinic connection problem (1.1)-(1.2).

The success of this strategy depends on some control on the behavior of the potential at infinity as well as near the wells, so minimizing sequences for problem (1.7) stay bounded, and to ensure the re-parametrization in the equipartition of energy parameter (for which (1.8) holds) is a diffeomorphism from \((0, 1)\) to \((-\infty, +\infty)\). More concretely, we require

(A1) The zero set of \( W \) consists of \( m \) distinct points \( p_1, \ldots, p_m \) so that \( W(p_1) = \ldots = W(p_m) = 0 \), and \( W > 0 \) elsewhere.

(A2) \( \lim \inf_{|p| \to \infty} W(p) > 0 \).

(A3) \( W \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{p_1, \ldots, p_m\}) \).

(A4) There exist \( C, \delta > 0 \) so that \( W(p) \leq C|p - p_j|^2 \) for every \( p \in \bigcup_{j=1}^m B_\delta(p_j) \).

Let us now give a statement of the main result of Chapter 2

**Theorem 1.1.** Given \( m, N \geq 2 \) suppose \( W : \mathbb{R}^N \to [0, \infty) \) is a potential satisfying (A1) through (A4).

Then any pair of wells \( p_j, p_k \) can be joined with a curve \( \gamma_* \) which minimizes the length \( \ell_W \) amongst competitors having fixed endpoints \( p_j \) and \( p_k \). In addition,

(i) If \( \gamma_* \) does not go through other wells at intermediate times, then there is of this curve \( U(t) := \gamma_*(x(t)) \) solving the heteroclinic connection problem (1.1)-(1.2).

(ii) If \( \gamma_* \) visits at least three wells, say \( \gamma_*(t_1) = p_{i_1}, \gamma_*(t_2) = p_{i_2}, \ldots, \gamma_*(t_J) = p_{i_J} \) (for \( J \geq 3 \)) as the parameter increases, then every pair of consecutive wells \( \gamma_*(t_k) \) and \( \gamma_*(t_{k+1}) \) can be joined with a heteroclinic connection, obtained as a reparametrization of the restriction \( \gamma_*|_{(t_k, t_{k+1})} \).
For a precise statement of the result, see Theorem 2.3 and Theorem 2.1.

Let us observe some important consequences of this result. First of all, we can reinterpret the above theorem to give a sufficient geometric condition, in arbitrary dimensions $N \geq 2$, for the existence of heteroclinic connections between two wells of a potential having any number of wells.

**Corollary 1.1.** If the curve $\gamma_*$ of minimum length $\ell_W$ in Theorem 1.1 connecting two wells $p_j$ and $p_k$ of $W$ satisfies the strict triangle inequality

$$\ell_W(\gamma_*) = d_W(p_j, p_k) < d_W(p_j, p_l) + d_W(p_l, p_k) \quad \text{holds for all } l \notin \{j, k\}, \quad (1.9)$$

where $d_W$ is defined via (1.7), then under an equipartition parametrization, this curve represents an $H$-minimizing connection between the two wells.

The sufficiency of the strict inequality (1.9) had already been noted in [3] to yield existence of heteroclinic connections in the planar setting $N = 2$ (cf. (1.14) in [3]). A decade later, the authors in [9] further established (1.9) as a necessary condition for the existence of $\mathbb{R}^2$-valued heteroclinic connections between $p_j$ and $p_k$. As it turns out, in the planar case, heteroclinic connections need necessarily to be of minimizing character, that is to say, all critical points of $H$ need to solve (1.6). The rigidity of this striking result is surely related to the low-dimensionality of the space considered, $\mathbb{R}^2 \simeq \mathbb{C}$, which has been exploited through the use of complex-analytic techniques. We refer the reader to [8] for details of the characterization of heteroclinic connections using complex-variable theory. On the other hand, there are examples in higher dimensions $N \geq 3$ of heteroclinic connections which are not $H$-locally minimizing (see discussion at the end of Chapter 2), thus the rigidity in [9] is lost and (1.9) cannot be a necessary condition for existence of heteroclinic connections, in full generality.

Next, we obtain a positive answer for the existence of a vector-valued heteroclinic connection for an arbitrary two-well potential.
Corollary 1.2 (Two-well case). Assume $W$ vanishes exactly at two points $p_-$ and $p_+$. Then there exists an $H$-minimizing heteroclinic connection between these wells.

In comparing Corollary 1.2 to the earlier results found in [79] and [9], we point out that our assumptions (A1)-(A4) are quite weak. For example, in [79] there is an assumption that the Hessian matrix of $W$ is positive definite at the two wells, while in [9], the authors assume a radial monotonicity condition holds at the two wells of the form

$$\exists r_0 > 0 \text{ so that for all } \xi \in S^{N-1}, r \mapsto W(p_{\pm} + r\xi) \text{ is increasing for } r \in (0, r_0),$$

(1.10)

cf. assumption (h) in [9]. However, Corollary 1.2 above holds even for potentials that oscillate near the wells, such as

$$W_0(p) = \prod_{j=1}^{2} (2 + \sin(1/|p - p_j|))|p - p_j|^2 \text{ for } p \in \mathbb{R}^N, \text{ given } p_1, p_2 \in \mathbb{R}^N,$$

that do not satisfy assumption (h), see Figure 1.2 below.

Figure 1.2: Plot of oscillatory potential $W_0$ for the choice $p_1 = (0,0), p_2 = (1,0)$.

On the other hand, in the recent work [22] existence is obtained for a $C^1$ potential satisfying our assumptions (A1)-(A3) without (A4), and in [76] the author dispenses with the monotonicity assumption (1.10).

After completion of this research project, we learned about the reference [52] in which a similar result is established under a slightly milder assumption on the behavior of the potential $W$ at
infinity. The method of proof takes essentially the same length space perspective as in the present investigation. Our presentation here, however, is somewhat more thorough and self-contained.

We now describe in more detail the second problem of the first part of this thesis, namely, topic (II) as stated in the beginning of the introduction. This time we focus on describing additional properties enjoyed by heteroclinic connections solving systems of gradient type (1.1) in arbitrary dimensions \( N \geq 2 \), the same ones considered in Chapter 2. The motivation behind the study of the next problem comes from a question raised by two mathematicians, Francesca Alessio and Piero Montecchiari, who are experts in the study of systems of differential equations.

Let us restrict our attention to a smaller class of \( C^3 \)-potentials \( W : \mathbb{R}^N \to [0, \infty) \), having only two wells \( p_- \) and \( p_+ \) at value zero, in such a way that they are isolated critical points of \( W \), that is \( \{ W = c \} \subset \{ \nabla W \neq 0 \} \) for \( c > 0 \) small, and assume the potential satisfies a quadratic non-degeneracy at the wells, i.e. \( D^2 W(p_{\pm}) > 0 \). Moreover, provided \( W \) grows at infinity, we can make a reasonable assumption \((W_c)\) on the topology of the sublevel sets \( \{ W \leq c \} := \{ u \in \mathbb{R}^N : W(u) \leq c \} \), for values of \( c \in (0, 1) \) small enough:

\[ (W_c) \] There exist closed, bounded sets \( W^-_c, W^+_c \subset \mathbb{R}^N \) with \( p_- \in int(W^-_c), \ p_+ \in int(W^+_c) \), and such that \( \{ W \leq c \} = W^-_c \cup W^+_c \), while \( W^-_c \cap W^+_c = \emptyset \).

Loosely put, the problem consists of characterizing minimizing heteroclinic connections between \( p_- \) and \( p_+ \) as limits, in a suitable sense, of sequences of periodic vector-valued solutions to the above gradient system (1.1), each one bridging the two portions \( W^-_c \) to \( W^+_c \). Each periodic orbit, also known as a brake type orbit at level \( c \), bounces back and forth between two components of a the \( c \)-level set of the potential. The limit under which the brake type orbits converge to a heteroclinic connection corresponds to level set values tending to the minimum value of the potential.

We now present some background needed for a precise formulation of the main result in this direction. The existence of such periodic orbits has been established by Alessio and Montecchiari in an unpublished manuscript [6]. Their main result shows, upon requiring further assumptions on
\[ \frac{d^2}{dt^2} W(q(t)) \quad \text{in} \quad (-\infty, +\infty), \]  

satisfying a pointwise mechanical energy constraint at level \(-c\), namely,

\[ E_q := \frac{1}{2} |q'(t)|^2 - W(q(t)) = -c \quad \text{for all} \quad t \in \mathbb{R}. \]  

This orbit can be obtained by finding a minimizer of the following variational problem “at level \(c\),

\[ m_c = \inf \{ H_c(q) : q \in \mathcal{M}_c \}, \]  

where \( H_c \) is a vector-version of the Modica-Mortola energy, renormalized to level \(c\),

\[ H_c(q) := \int_{-\infty}^{+\infty} \frac{1}{2} |q'(t)|^2 + (W(q(t)) - c) \, dt, \]  

and the admissible class \( \mathcal{M}_c \) restricts the search to curves that make \( H_c \) well-defined and bounded from below by zero, and whose orbits bridge the components of the \(c\)-sublevel set of \(W\). The analysis performed in [6] of the variational problem (BTPc) reveals that there exists a minimizer \(q_c \in \mathcal{M}_c\) satisfying \(q_c(\mathbb{R}) \subset \{ W \geq c \}\), and so that there are distinguished times \(-\infty < \overline{\alpha}_c < \overline{\omega}_c < +\infty\) for which

\[ H_c(q_c) = \int_{\overline{\alpha}_c}^{\overline{\omega}_c} \frac{1}{2} |q'_c(t)|^2 + (W(q_c(t)) - c) \, dt \]  

= \( H_{(\overline{\alpha}_c, \overline{\omega}_c)}(q_c) \).  

Since the integrand in \( H_c(q) \) is always non-negative for \(q \in \mathcal{M}_c\), we conclude from (1.14) that \(q_c(t) \equiv q_c(\overline{\alpha}_c)\) for all \(t \leq \overline{\alpha}_c\) and \(q_c(t) \equiv q_c(\overline{\omega}_c)\) for all \(t \geq \overline{\omega}_c\). This fact combined with the constraint on the mechanical energy (1.12) yields the important identity

\[ H_c(q_c) = 2 \int_{\overline{\omega}_c}^{\overline{\omega}_c} (W(q_c) - c). \]  

Furthermore, it can be argued that \(q_c(\overline{\alpha}_c) \in W^-_c\) and \(q_c(\overline{\omega}_c) \in W^+_c\), so we say that these points are contact points of the trajectory \(q_c\) with \(W_c^\pm\), and that \(\overline{\alpha}_c, \overline{\omega}_c\) are contact times.

**Note.** We have intentionally adopted a cumbersome notation for \(q_c, \overline{\alpha}_c\) and \(\overline{\omega}_c\) because we have reserved \(q, \alpha_c\) and \(\omega_c\) for a corresponding renormalization carried out in \(\S 3.2.2\).
Let us observe that if \( t \) is a contact time then \( W(q_c(t)) \leq c \), then the energy condition (1.12) imposes that \( W(q_c(t)) = c \) and \( q'_c(t) = 0 \). Hence, \( t \) can be considered a turning time, i.e., a time at which the orbit \( q_c \) is symmetric with respect to \( t \). This argument applies for any such contact time, including \( t = \overline{\alpha}_c \) and \( t = \overline{\omega}_c \). Thus a solution \( q \) to (1.11) satisfying (1.12) is built periodically by reflecting the minimizer of (BTPc) at the contact times \( \overline{\alpha}_c \) and \( \overline{\omega}_c \).

\[
q(t) = \begin{cases} 
q_c(2\overline{\alpha}_c - t) & \text{for } t \in (2\overline{\alpha}_c - \overline{\alpha}_c, \overline{\alpha}_c], \\
q_c(t) & \text{for } t \in (\overline{\alpha}_c, \overline{\omega}_c), \\
q_c(2\overline{\omega}_c - t) & \text{for } t \in [\overline{\omega}_c, 2\overline{\omega}_c - \overline{\alpha}_c),
\end{cases}
\tag{1.15}
\]

and the definition carries over in a similar fashion over the remaining intervals. Hence \( q \) oscillates back and forth in the configuration space along the arc \( q_c([\overline{\alpha}_c, \overline{\omega}_c]) \), it verifies \( W(q_c(t)) > c \) for any \( t \in (\overline{\alpha}_c, \overline{\omega}_c) \), and it has period \( 2(\overline{\omega}_c - \overline{\alpha}_c) \). Such a solution constructed as in (1.15) is said to be a brake type orbit.

The general scheme of finding solutions with prescribed energy to second order differential equations, that connect two components of a set, has already been explored in [4]. The author studies existence and qualitative behavior of solutions to the Allen-Cahn type system

\[
-\Delta u(x, y) + \nabla W(u(x, y)) = 0 \quad \text{for } (x, y) \in \mathbb{R}^2,
\tag{1.16}
\]

for a non-negative two-well potential \( W \), symmetric with respect to \( x \). Provided the set of one-dimensional heteroclinic connections (solving (1.11)) that join the wells of \( W \) has a discrete structure (condition (*) in [4]), the author is able to construct infinitely many layered solutions to (1.16) satisfying the asymptotics \( \lim_{x \to \pm \infty} u(x, y) = p_\pm \) uniformly in \( y \in \mathbb{R} \). These correspond to an infinite dimensional analogue of (1.11)-(1.12), in that, these solutions satisfy

\[
\frac{d^2}{dy^2} u(\cdot, y) = \delta H^0(u(\cdot, y)),
\]

with \( H^0 \) as defined in (1.13) above and where \( \delta H^0 \) stands for the first variation of the functional \( H \) (we note that \( H^0 \) is the same as the functional \( H \) introduced earlier). In addition, these solutions
satisfy \( E_u := \frac{1}{2} \| \partial_y u(\cdot, y) \|_{L^2}^2 - H^0(u(\cdot, y)) \equiv -c \) for all \( y \in \mathbb{R} \); and \( \lim_{y \to \pm \infty} \text{dist}_{L^2}(u(\cdot, y), V_c^\pm) = 0 \) where the sets \( V_c^\pm \) form a partition of a certain subset of \( \{ q \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) : H^0(q) \leq c \} \) with \( \text{dist}_{L^2}(V_c^-, V_c^+) > 0 \). The author builds bidimensional heteroclinic type, homoclinic type and brake type solutions to the Lagrangian system (1.11) connecting \( V_c^- \) to \( V_c^+ \). For more details cf. [4, Theorem 1.2].

The main goal of Chapter 2 is to establish a convergence result for a sequence of brake type orbits \( \{ q_c \} \), as \( c \to 0^+ \), each one solving (3.1) and bridging \( \{ W \leq c \}^- \) to \( \{ W \leq c \}^+ \), to some classical solution \( U_0 \) of the heteroclinic connection problem between the wells of the potential,

\[
U_0'' = \nabla W(U_0) \quad \text{in} \quad \mathbb{R},
\]

\[
U_0(-\infty) = p_-, \quad U_0(+\infty) = p_+.
\]

This limiting solution \( U_0 \) turns out to be a minimizer of the heteroclinic variational problem

\[
m_0 := \inf \{ H^0(V) : V \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \}, \quad \text{(HCP)}
\]

where the admissible class is an affine Sobolev space

\[
H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) := U_{\text{aff}} + H^1(\mathbb{R}; \mathbb{R}^N),
\]

and \( U_{\text{aff}} \) is a linear affine function connecting the two wells, see (3.10) in Chapter 3. The existence of such an \( H^0 \)-minimizing heteroclinic is guaranteed under reasonable assumptions on the potential, to be made precise later, by one of the main results in Chapter 2 since \( W \) is a double-well potential (see Corollary 1.2).

One can formally regard the heteroclinic variational problem (HCP) as a limit when \( c \to 0^+ \) of the variational problems (BTPc) which give rise to brake type orbits. The primary goal of this chapter consists of a rigorous study of the relationship between these two variational problems.

In order to carry out the analysis one needs, in some sense, to extend the trajectories of brake type orbits up to the wells of \( W \), i.e. the components of the zero-level set of \( W \). In §3.3 we devise a method that for small values of \( c \) allows us to relate curves in \( \mathcal{M}_c \), admissible for the
variational problem (BTPc) “at level c”, with curves in $H^1_{\text{aff}}(\mathbb{R};\mathbb{R}^N)$ admissible for the heteroclinic problem (HCP) “at level zero”. Given $c > 0$ and a minimizer $q_c$ of (BTPc), we create a new curve $q_{c,2c} \in H^1_{\text{aff}}(\mathbb{R};\mathbb{R}^N)$ by modifying $q_c$ is the following manner: The set of times where $q_c$ lies on the superlevel set $\{ W > 2c \}$ will be shown to be a bounded open interval that we can denote $(\tau^-_{2c}, \tau^+_{2c})$.

The new curve $q_{c,2c}$ equals $q_c$ on $(\tau^-_{2c}, \tau^+_{2c})$, however on $(-\infty, \tau^-_{2c})$ and $(\tau^+_{2c}, +\infty)$ we define it as the solution to the corresponding gradient flow initial value problem

$$\pm \Phi'(t) = -\nabla W(\Phi(t)) \quad \text{for} \quad \pm t \geq 0, \quad \Phi(0) = q_c(\tau^{\pm}_{2c}).$$

In this way, we have effectively constructed a curve that connects the wells $p_-$ and $p_+$, with finite energy $H^0$. For details we refer the reader to the definitions in §3.3.

**Remark 1.1.** The “natural” extension of the minimizer $q_c : [\alpha_c, \omega_c] \rightarrow \mathbb{R}^N$ to the wells using gradient flow, starting from contact times at $\{ W = c \}^{\pm}$, has not shown to be as effective as the given one, mainly due to technical difficulties in our analysis.

In order to make this procedure work, including the very existence of the family of minimizers $\{q_c\}$, we need additional hypotheses on the potential. We emphasize, however, that most of these are standard in the literature.

(W1) $W \in C^3(\mathbb{R}^N)$ and $p_\pm$ are the only global minima at zero, $W(p_\pm) = 0$. Furthermore,

$$\lim_{|p| \to \infty} \inf W(p) > 0.$$

This assumption, among other things, ensures that the value $m_c := \inf \{ H^c(q) : q \in \mathcal{M}_c \}$, for sufficiently small $c > 0$, does not increase when minimizing over a reduced admissible set with $L^\infty$-constraint. In other words, there is $R_0 > 0$ so that

$$m_c = \inf \{ H^c(q) : q \in \mathcal{M}_c, \|q\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} \leq R_0 \}.$$ 

(W2) There exist $0 < \lambda \leq \Lambda < \infty$ so that

$$\lambda I_{N \times N} \leq D^2 W(p_\pm) \leq \Lambda I_{N \times N}.$$
Let us denote the linear segment in $\mathbb{R}^N$ between the points $p_1$ and $p_2$ by

$$[p_1, p_2] := \{p_\lambda \in \mathbb{R}^N : p_\lambda = (1 - \lambda)p_1 + \lambda p_2 \text{ for some } \lambda \in [0, 1]\}.$$ 

Suppose in addition that there exists $c_0 \in (0, \max[p_-, p_+] W)$ in such a way the following properties hold

(W3) $\forall c \in (0, c_0] : \{W \leq c\}$ is partitioned into disjoint sets $W_c^- = \{W \leq c\}^-, W_c^+ = \{W \leq c\}^+$, each enclosing $p_-$ and $p_+$, respectively.

(W4) Every $c \in (0, c_0]$ is a regular value for $W$, i.e., $\bigcup_{c \in (0, c_0]} \{W = c\} \subset \{\nabla W \neq 0\}$.

Now we are ready to introduce the last main result of Part I in Chapter 3:

**Theorem 1.2.** Assume $W$ satisfies (W1) through (W4). Then for any sequence $c_n \to 0^+$, the family of variational problems (BTP$_{c_n}$) approaches the variational heteroclinic connection problem (HCP), in the following senses:

For any sequence $\{q_{c_n}\}$ of minimizers to (BTP$_{c_n}$), there holds

(i) $\lim_{n \to \infty} m_{c_n} = m_0$, where $m_{c_n} := H^{c_n}(q_{c_n})$ and $m_0 := \inf \{H^0(V) : V \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)\}$.

(ii) There exist times $\{t_k\} \subset \mathbb{R}$, and a subsequence $\{q_{c_n_k, 2c_n_k}\}$ of the gradient flow extensions from the level $2c_n_k$ of the minimizers $\{q_{c_n}\}$ (see Definition 3.3) such that for $q_k := q_{c_n_k, 2c_n_k}(\cdot - t_k)$ one has

$$q_k - U_0 \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}; \mathbb{R}^N) \text{ as } k \to \infty,$$

where $U_0$ is a minimizer of (HCP), in particular, a heteroclinic connection between the wells of $W$.

**Remark 1.2.** The convergence (1.17) stated in the previous Theorem 1.2, is sharp in the sense that the translation in time of the sequence of brake orbits $\{q_c\}$ cannot be avoided. This is due to the translation invariance that the energies $H^c$ and the gradient system (1.11) satisfy. See the paragraph following (1.6) for a discussion of this issue.
Remark 1.3. Using the embedding $H^1(\mathbb{R}; \mathbb{R}^N) \hookrightarrow C^{0,1/2}(\mathbb{R}; \mathbb{R}^N)$ and a diagonalization argument, one can deduce from the convergence (1.17) the additional types of convergence of the sequence \( \{q_k\} \) to a heteroclinic connection:

\[
\begin{align*}
q_k & \xrightarrow{k \to \infty} U_0 \quad \text{in} \quad C^{0,\alpha}_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \quad \text{for any} \quad \alpha \in (0, 1/2), \quad \text{and} \\
q_k & \xrightarrow{k \to \infty} U_0 \quad \text{a.e. on} \quad \mathbb{R}.
\end{align*}
\]

See the proof of Theorem 3.1 for details.

Remark 1.4. The extension of the brake type orbits, \( \{q_{c_n k, 2 c_n k}\} \), need not be carried out using the gradient flow of \( W \) necessarily (see §3.3.3 for details of the latter definition). In fact, minor modifications to the proof of Theorem 1.2 show that the orbit \( q_c \) can be extended by any pair of curves \( \Psi_1 = \Psi_1(\cdot, b^-) : (-\infty, 0] \to \mathbb{R}^N \) and \( \Psi_2 = \Psi_2(\cdot, b^+) : [0, +\infty) \to \mathbb{R}^N \), bridging any given initial data \( b^\pm \in \{W \leq c_0\}^\pm \) to the wells \( p_\pm \), provided that \( \Psi_1, \Psi_2 \) do not contribute \( H^0 \)-energy asymptotically as \( W(b^\pm) \to 0^+ \), and provided that \( \Psi_1, \Psi_2 \) stay trapped in small level sets of the potential. In other words, \( \Psi_1, \Psi_2 \) must satisfy

- For some function \( f : (0, c_0) \to (0, \infty) \) with \( \lim_{x \to 0^+} f(x) = 0 \), there holds

\[
\int_{-\infty}^0 \frac{1}{2} |\Psi_1'(t, b^-)|^2 + W(\Psi_1(t, b^-)) \, dt \leq f(W(b^-)) \quad \text{for all} \quad b^- \in \{W \leq c_0\}^-, \tag{1.19}
\]

\[
\int_0^{+\infty} \frac{1}{2} |\Psi_2'(t, b^+)|^2 + W(\Psi_2(t, b^+)) \, dt \leq f(W(b^+)) \quad \text{for all} \quad b^+ \in \{W \leq c_0\}^+. \tag{1.20}
\]

- The following pointwise bound holds

\[
W(\Psi_1(t, b^-)) \leq W(b^-) \quad \text{for all} \quad t \leq 0, \quad \text{and all} \quad b^- \in \{W \leq c_0\}^-, \tag{1.20a}
\]

\[
W(\Psi_2(t, b^+)) \leq W(b^+) \quad \text{for all} \quad t \geq 0, \quad \text{and all} \quad b^+ \in \{W \leq c_0\}^+. \tag{1.20b}
\]

(cf. Corollary 3.1 and Lemma 3.2 in the case of gradient flow extensions).

In particular, if one is willing to assume further hypotheses on the geometry on level sets of \( W \) near the wells, one can extend the brake type orbits using special choices of \( \Psi_1, \Psi_2 \). For example, if the components of small level sets are star-shaped with respect to each corresponding well, namely, the condition below holds
(S_c) For all \( p \in \{ W \leq c \}^\pm \), \([p_\pm, p] := \{ \lambda p + (1 - \lambda)p_\pm : \lambda \in [0, 1] \} \subset \{ W \leq c \}^\pm \), for all \( c \) small enough, then (1.19)-(1.20) are satisfied by linear-type extensions to the wells:

\[
\Psi_j(t, b^\pm) = \lambda(t)b^\pm + (1 - \lambda(t))p_\pm, \quad j = 1, 2,
\]

provided \( \lambda \in H^1(\mathbb{R}) \cap C^1_b(\mathbb{R} \setminus \{0\}) \) satisfies \( \lambda(\mathbb{R}) = (0, 1] \), with \( \lambda(0) = 1 \), and \( \lambda \) is monotone increasing on \((-\infty, 0)\) and monotone decreasing on \((0, +\infty)\).

In a very closely related work [41], the author establishes a result saying that any minimizing sequence of \( \text{(HCP)} \) converges, in the sense of (1.17) and up to translates in time, to a minimizing heteroclinic connection \( U_0 \). In other words, translation invariance causes the only possible loss of compactness to minimizing sequences. However, we point out that he assumes convexity of the level sets of \( W \) near the wells, cf. \((S_c)\). This is a rather stringent hypothesis, that we have managed to remove; in fact, we assume no geometry of the small level sets of \( W \) whatsoever. In part, this is due to the fact that the scope of the result in this thesis is limited to the specific minimizing sequences constructed out of brake type orbits. The general strategy in the proof of Theorem 1.2 to retrieve compactness of the sequence of brake type orbits, however, has been largely influenced by the techniques in [41].

Lastly, we now describe the problem of the second part of this thesis corresponding to topic (III) as initially stated. Our primary focus of interest are functions \( u : \Omega \subset \mathbb{R}^N \to \mathbb{R} \) that minimize a weighted version (or inhomogeneous) of the total variation functional

\[
\int_U a(x)|Du| \quad \text{(aTVF)}
\]

subject to Dirichlet boundary data \( u = g \) on \( \partial \Omega \), for a reasonable class of domain \( \Omega \). Here the weight function \( a \) is continuous and bounded away from zero, and arbitrary dimensions \( N \geq 2 \) are considered. Any such minimizer will be called a function of weighted least gradient, and if \( a \equiv 1 \) then we refer to such a minimizer simply as a function of least gradient.
The total variation functional in the classical case $a \equiv 1$,

$$
\int_{\Omega} |Du|
$$

is closely related to the theory of minimal surfaces. Formally, critical points of (1.21) in a suitable admissible class weakly solve

$$
0 = \text{div} \left( \frac{Du}{|Du|} \right) =: -H \quad \text{in } \Omega,
$$

where $H$ corresponds to the mean curvature of the level sets of $u$, $\{ x \in \Omega : u(x) = c \} \subset \mathbb{R}^N$, provided $u$ is regular enough. This is related to the so-called Plateau problem in the theory of minimal surfaces (see e.g. [30] for a full treatment on the subject). In this regard, a rigorous analysis has been carried out in [46] to characterize functions of least gradient minimizing (1.21) subject to a Dirichlet boundary condition $g \in L^1(\partial \Omega)$ as solutions to the associated Dirichlet boundary-value problem of the above Euler-Lagrange equation (1.22)

$$
-\text{div} \left( \frac{Du}{|Du|} \right) = 0 \quad \text{in } \Omega,
$$

$$
u = g \quad \text{on } \partial \Omega,
$$

in a suitable sense, to handle the case where $Du$ vanish inside of $\Omega$.

The functional in (1.21) formally corresponds to a limit as $p \to 1^+$ (in the singular regime $1 < p < 2$) of the classical $p$-Dirichlet functional in the calculus of variations

$$
\int_{\Omega} \frac{1}{p} |Du|^p \, dx.
$$

By minimizing (1.23) in a class of functions satisfying a Dirichlet boundary-condition, $\{ u \in W^{1,p}(\Omega) : u = g \text{ in the sense of traces} \}$, one obtains a unique weak solution to the $p$-Laplace equation

$$
\Delta_p u := \text{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega,
$$

$$
u = g \quad \text{on } \partial \Omega.
$$

Indeed is the case that the variational problems obtained by minimizing (1.21) and (1.23) are strongly related. Juutinen in [40] showed that functions of least gradient (minimizing the former
problem) can be characterized as uniform limits as \( p \to 1^+ \) of \( p \)-harmonic functions (minimizing the latter problem).

Let us now observe that \((aTVF)\) simply corresponds to
\[
\int_{\Omega} a(x) |\nabla u(x)| \, dx,
\]
for functions that are regular enough, say \( u \in C^1(\bar{\Omega}) \), and moreover, this expression (1.24) works as well as a definition of \((aTVF)\) for functions belonging to the Sobolev space \( u \in W^{1,1}(\Omega) \), where \( \nabla u \) now refers to the weak gradient of \( u \) in \( \Omega \). More generally, \((aTVF)\) is defined over the larger class \( BV(\Omega) \) of functions of bounded variation in \( \Omega \), see e.g. [25,30]. Any such function \( u \) is integrable, i.e. \( u \in L^1(\Omega) \), and admits first derivatives \( Du \) that are to be interpreted as a vector-valued Radon measure in \( \mathbb{R}^N \), acting on test functions by
\[
\int_{\Omega} \psi \, dDu = - \int_{\Omega} u(x) \nabla \psi(x) \, dx \in \mathbb{R}^N, \quad \forall \psi \in C^1_c(\Omega).
\]
Originally introduced by Amar and Bellettini in [10], the \( a \)-variation \((aTVF)\) of \( u \in BV(\Omega) \) in the case the weight is continuous and bounded away from zero is equivalent to
\[
\int_{\Omega} a \, d|Du|, \tag{1.25}
\]
where \( |Du| \) stands for the total variation measure of \( Du \). However, the authors in [10] argue that (1.25) cannot be used to replace definition of \((aTVF)\) if the weight \( a \) fails to be continuous or bounded below. See formula (4.4) in Chapter 4 for the more general defining expression of the \( a \)-total variation functional.

The functional \((aTVF)\) gives rise to a Radon measure on \( \mathbb{R}^N \) that acts on Borel sets via \( B \mapsto \int_{B} a(x)|Du| \), commonly known as the \( a \)-variation measure of \( u \). By analogy, one can define the \( a \)-perimeter (or \( a \)-area) measure of a set of finite perimeter \( E \subset \mathbb{R}^N \) by
\[
\mathcal{P}_a(E, B) := \int_{B} a(x)|D\chi_E| \quad \text{for } B \text{ Borel,} \tag{1.26}
\]
where \( \chi_E \) is the characteristic function of \( E \). For sets \( E \) whose boundary are regular enough, \( \partial E \in C^2 \), one can identify (1.26) with the usual \((N - 1)\)-perimeter measure of \( \partial E \) inside \( B \) having density \( a \); that is, \( \mathcal{P}_a(E, B) = \int_{\partial E \cap B} a(x) d\mathcal{H}^{N-1} \), where \( d\mathcal{H}^{N-1} \) is the \((N - 1)\)-dimensional Hausdorff measure, i.e. surface measure.

Let us now introduce the problem of Chapter 4 in more detail. Given a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^N \) in some dimension \( N \geq 2 \), and a continuous weight function \( a : \bar{\Omega} \to \mathbb{R} \) satisfying the following non-degeneracy condition
\[
\min_{\bar{\Omega}} a \geq \alpha > 0,
\]
for some \( \alpha \in (0, \infty) \), we are interested in the study of minimizers of the weighted \( a \)-variation functional amongst functions of bounded variation that coincide on the boundary with given data \( g : \partial \Omega \to \mathbb{R} \). That is, we deal with the variational problem
\[
\inf_{u \in BV_g(\Omega)} \int_{\Omega} a(x)|Du|,
\]
where the admissible class is defined as
\[
BV_g(\Omega) := \{ u \in BV(\Omega) : u = g \text{ on } \partial \Omega \text{ in the sense of traces} \}.
\]
Under the above assumptions on \( u \) and \( \Omega \) we have the validity of the co-area formula
\[
\int_{\Omega} a(x)|Du| = \int_{-\infty}^{+\infty} \mathcal{P}_a(\{u \geq t\}, \Omega) dt,
\]
where \( \{u \geq t\} := \{ x \in \mathbb{R}^N : u(x) \geq t \} \) denotes the superlevel set of \( u \) in \( \mathbb{R}^N \), see e.g. [10].

This identity has important geometric implications. In principle, one could construct a minimizer of \((aLGP)\) by first solving the auxiliary problem, given \( t \in \mathbb{R} \), of finding a set \( E_t \) of minimal \( a \)-perimeter \( \mathcal{P}_a(\cdot, \Omega) \) among sets whose boundaries in \( \partial \Omega \) meet \( \{g = t\} \). Through a gluing procedure one could then define \( u \) “level set by level set”, so as to assign the value \( u(x) \) depending on the collection of sets \( \{E_t\} \) that \( x \) lies in, as the values of \( t \in \mathbb{R} \) vary. It is precisely this strategy that we will adopt for a constructive method to find solutions to \((aLGP)\).
This use of the co-area formula has a long history, and it lies at the foundation of the theory of minimal surfaces. In the seminal work [18], Bombieri, De Giorgi and Giusti exploited (1.29) to show that continuous functions of least gradient (i.e., solving (aLGP) in the case $a \equiv 1$) have their superlevel sets minimizing the usual notion of perimeter (or area), and in particular their boundaries constitute minimal surfaces. Conversely, in the same setting $a \equiv 1$ Sternberg, Williams and Ziemer proved in [80] the existence and continuity of a function of least gradient, by means of carrying out a constructive analysis of the strategy outlined in the preceding paragraph.

While the interest in functions of least gradient dates back to the work in [18], the literature on the subject has been expanding throughout the years up to this date, where investigations are carried out in a variety of different contexts. The majority of the existing results for least gradient problems study the case of Dirichlet boundary conditions (see for instance [31,39,47,57]). Nonetheless, Neumann and other types of boundary conditions have been explored (cf. [55,61,64]). In the recent years many authors have spent a significant effort to study weighted least gradient problems and further generalizations, due to its various applications to such areas as current density impedance imaging, reduced models in superconductivity and superfluidity, models for a description of landsliding, and relaxed models in the theory of elasticity and in optimal design, among others. A list of important investigations in these directions can be found in [13,14,31,32,37–39,42,47,54–57,61–65,77,80,81]. In addition, the time dependent notion of total variation flow has proved to be useful in image processing including denoising and restoration, see for example [12,15,23,26,51]. Further generalizations of least gradient problems in the metric space setting have been explored quite recently in [35,43,44].

Regarding the study of least gradient problems subject to Dirichlet boundary condition, in one of the first investigations in the subject, [80], Sternberg, Williams and Ziemer explored many aspects of the standard least gradient problem

$$\inf \left\{ \int_\Omega |Du| : u \in BV(\Omega), \ u = g \ \text{on} \ \partial\Omega \right\}, \quad (1.30)$$
where continuous boundary data \( g : \partial \Omega \to \mathbb{R} \) are considered. First they establish existence and continuity of a function of least gradient for every dimension \( N \geq 2 \), by explicitly constructing each of its superlevel sets in such a way that they are area-minimizing and reflect the boundary condition. Secondly, they prove a comparison theorem for functions of least gradient with respect to their boundary-data, which in particular implies uniqueness of continuous solutions to (1.30). Thirdly, they study higher regularity properties of functions of least gradient; if the boundary data \( g \) is more than merely continuous, one may expect solutions to reflect this additional regularity.

Parks had already proved in [66] the existence of Lipschitz solutions, provided a bounded slope condition is satisfied. On the other hand, the examples in [67] show that, in general, Lipschitz continuity is the most that can be expected for a function of least gradient. The regularity theory was further developed by the authors in [80] by showing that if \( \partial \Omega \) is strictly mean convex and the Dirichlet boundary data is Hölder continuous \( g \in C^{0,\alpha}(\partial \Omega) \) for some \( \alpha \in (0,1) \), then the solution to (1.30) is Hölder continuous as well, \( u \in C^{0,\alpha/2}(\Omega) \). In the same paper they argue that this Hölder modulus of continuity \( \alpha/2 \) is sharp.

Their proof of existence and continuity restricts the class of admissible domains to those whose boundary \( \partial \Omega \) satisfies two geometric conditions, referred as a weak non-negative mean curvature condition and the assumption that \( \partial \Omega \) is not locally area-minimizing with respect to interior set variations. It has been pointed out by Spradlin and Tamasan in [77] that even for domains \( \Omega \) satisfying these two conditions, the continuity of the boundary data \( g \) is necessary to ensure the existence of minimizers to

\[
\inf \left\{ \int_\Omega |Du| : u \in BV(\Omega), \ u|_{\partial \Omega} = g \right\}.
\]

A secondary reason for the work in Chapter 4 has been to determine whether the Sternberg-Williams-Ziemer program in [80] carries over to the setting of weighted least gradient problems. In the context of Chapter 4, we adapt these two conditions to now require that \( \Omega \) is a bounded Lipschitz domain with connected boundary, which in addition fulfills the following
**Condition 1 (Barrier condition).** For every $x_0 \in \partial \Omega$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ if $V_\varepsilon \subset \Omega$ is a minimizer of

$$\inf \{ \mathcal{P}_a(W, \mathbb{R}^N) : W \subset \Omega, (\Omega \setminus W) \subset B_\varepsilon(x_0) \}, \quad (1.31)$$

then

$$\partial V_\varepsilon \cap \partial \Omega \cap B_\varepsilon(x_0) = \emptyset.$$ 

The boundaries of such domains $\Omega$ are not locally $a$-area minimizing with respect to interior variations. This condition had already been introduced in [39], where the authors point out that (1.31) is actually equivalent to the geometric conditions imposed to $\partial \Omega$ in [80] in the case $a \equiv 1$.

In the recent years an extensive study of a more general class of least gradient problems with Dirichlet boundary condition had been carried out by Jerrard, Moradifam and Nachman in [39]. The problem they consider is

$$\inf_{u \in BV_g(\Omega)} \int_{\Omega} \varphi(x, Du), \quad (\varphi_{DLGP})$$

for the admissible class given in (1.28), continuous boundary data $g : \partial \Omega \to \mathbb{R}$, and a function $\varphi(x, \xi)$ that, among other properties, is convex, continuous, and 1-homogeneous with respect to the $\xi$-variable. They establish existence and uniqueness results in all dimensions for minimizers of $(\varphi_{DLGP})$, which includes the present case ($aLGP$). Additionally, for boundary data $g_1, g_2 \in C(\partial \Omega)$ with $g_1 \geq g_2$ they prove a comparison result for the corresponding solutions $u_1, u_2$ to $(\varphi_{DLGP})$ in $\Omega$, in the same spirit as in [80], which yields uniqueness of solution to $(\varphi_{DLGP})$.

Furthermore, this unique solution was shown to be continuous in dimensions $N \leq 7$.

The dimensionality restriction in [39] is due to an appeal to the regularity theory of hypersurfaces minimizing parametric elliptic functionals of Almgren, Schoen and Simon [70, 71]. The major thrust of the work in Chapter 4 is to establish such a continuity result for a minimizer of (4.1) *in higher dimensions $N \geq 8$ as well*, even in the possible presence of singularities in the level sets of the minimizer. We recall the fact that for $N \geq 8$, area-minimizing hypersurfaces may contain
singularities, the most famous example being the celebrated Simons cone given by the formula

\[ \{ x = (y, z) \in \mathbb{R}^4 \times \mathbb{R}^4 : |y|^2 = |z|^2 \} \subset \mathbb{R}^8, \]

see [18] and [74] for details.

Quite different features on the models arise, for example, from the consideration of Neumann boundary conditions instead of Dirichlet-type, where a loss of uniqueness of solutions is to be expected. Moradifam in [55] considered

\[ \inf \left\{ \int_{\Omega} \varphi(x, Du) : u \in BV(\Omega), \int_{\partial \Omega} u = 0, \int_{\partial \Omega} ug = 1 \right\}, \quad (\varphi_{NLGP}) \]

where \( \varphi(x, \xi) \) enjoys the same properties as in (\( \varphi_{DLGP} \)), and \( g \in L^\infty(\partial \Omega) \). He established the existence of infinitely many minimizers to (\( \varphi_{NLGP} \)) for a given Neumann boundary data \( g \neq 0 \). In addition, he proved there exists a divergence-free vector field \( T \) that determines the structure of the level sets of all the minimizers, namely, that \( T(x) \) determines \( Du/|Du| \) for \( |Du| \)-a.e. \( x \in \Omega \) for every minimizer \( u \in BV(\Omega) \).

We now give the statement of the main result of Chapter 4.

**Theorem 1.3.** For any \( N \geq 2 \), let \( \Omega \subset \mathbb{R}^N \) be a bounded Lipschitz domain satisfying the barrier condition (1.31), and let \( a \in C^3(\bar{\Omega}) \) be a non-degenerate weight function in the sense of (1.27). Then for any boundary data \( g \in C(\partial \Omega) \), there exists a minimizer \( u \) to \( (aLGP) \), which is moreover a continuous function \( u \in C(\bar{\Omega}) \). Furthermore, the superlevel sets of this function minimize the weighted perimeter measure \( \mathcal{P}_a(\cdot, \Omega) \) with respect to competitors meeting the boundary conditions imposed by \( g \) on \( \partial \Omega \).

The main concern of this work will be to establish the continuity of the minimizer of \( (aLGP) \) even in the possible presence of **singularities for the level sets** of the solution, under the hypotheses of Theorem 1.3. This is a major technical point in the proof, and so it is worth explaining in some detail. To this end, one crucial ingredient in the proof of the desired continuity in Theorem 1.3,
as well as in [80], is establishing a strict separation of level sets. This property relies, among other things, on a strict maximum principle for area-minimizing sets established by Simon in [72] that will allow us to deal with the case of singularities. In our case we need to specifically adapt [72] in the context of weighted perimeter (or area). This will be accomplished in §4.3.2 by identifying the $\alpha$-perimeter of a set with the mass of its induced boundary current, in a Riemannian context:

$$M_{U, \alpha^{2/(N-1)}\bar{g}}(\partial[E]) = \mathcal{P}_\alpha(E, U),$$

where the current $\partial[E]$ corresponds to a generalized notion of hypersurface, $M$ stands for the mass of a current that is analogous to the concept of an operator norm, $(U, \alpha^{2/(N-1)}\bar{g})$ denotes an open set $U \subset \mathbb{R}^N$ seen as a Riemannian manifold endowed with a metric conformal to the standard Euclidean metric $\bar{g}_x = \delta_{ij}dx^i dx^j$ with conformal factor $\alpha^{2/(N-1)}(x)$, and $\mathcal{P}_\alpha$ is as defined in (1.26). For details we refer the reader to Theorem 4.4. As far as we know, this is the first time such a relation has been made explicit, at least in the literature regarding least gradient problems.
Part I

Study of heteroclinic connections of gradient systems

Chapter 2

On the heteroclinic connection problem for multi-well gradient systems

2.1 Introduction and statement of the main Theorem 2.1

The purpose of this chapter is to revisit the question of existence of heteroclinic connections associated with multiple-well potentials. Let us recall this problem from the introduction in Chapter 1: Given a potential $W : \mathbb{R}^N \rightarrow [0, \infty)$ whose zero set consists of $m$ distinct global minima $p_1, \ldots, p_m \in \mathbb{R}^N$, with $m \geq 2$, we pursue the question of existence of solutions $U : \mathbb{R} \rightarrow \mathbb{R}^N$ to the Hamiltonian system

$$U'' - \nabla_u W(U) = 0 \quad \text{in} \quad (-\infty, +\infty),$$

$$U(-\infty) = p_j, \quad U(+\infty) = p_k,$$

connecting any two of the wells $p_j, p_k$ with $j \neq k$.

The approach in [79] to establish the existence of heteroclinic connections, which has inspired [3] and ours as well, is first study the geometric problem of finding a minimizer of

$$d_W(p_1, p_2) := \inf \left\{ \int_{-\infty}^{+\infty} \sqrt{W(U)} |U'| : U(-\infty) = p_1, U(+\infty) = p_2 \right\},$$

(2.2)
for the case of two wells when \( U \) is \( \mathbb{R}^2 \)-valued \((m = 2 \text{ and } N = 2)\). The author studies (2.2) by solving the perturbed problem

\[
\inf E_\delta(U) \quad \text{with} \quad E_\delta(U) := \int_\mathbb{R} \left( \sqrt{W(U)} + \delta \right) |U'|,
\]

for \( \delta > 0 \) in which the degeneracy is removed, and then passing to the limit \( \delta \to 0 \) in the minimizers. Obtaining \( \delta \)-independent bounds to establish the needed compactness for this procedure, however, is somewhat messy and seems to require rather strong assumptions on the non-degeneracy of the Hessian of \( W \) at the wells. In the present approach in this chapter, we work directly in the metric space \( \mathbb{R}^N \) endowed with metric \( d_W \) and obtain much more general results with far weaker hypotheses. This will allow us to consider the following class of potentials \( W : \mathbb{R}^N \to \mathbb{R} \) satisfying

(A1) The zero set of \( W \) consists of \( m \) distinct points \( p_1, \ldots, p_m \) so that \( W(p_1) = \ldots = W(p_m) = 0 \), and \( W > 0 \) elsewhere.

(A2) \( \liminf_{|p| \to \infty} W(p) > 0 \).

(A3) \( W \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{p_1, \ldots, p_m\}) \).

(A4) There exist \( C, \delta > 0 \) so that \( W(p) \leq C|p - p_j|^2 \) for every \( p \in \bigcup_{j=1}^m B_\delta(p_j) \).

In §2.2 we pursue the question of existence of minimizers of (2.2) under very mild assumptions on the multi-well potential \( W \), basically just continuity and non-zero behavior at infinity (A2). We first introduce a notion of length \( \ell_W \) induced by the metric space \((\mathbb{R}^n, d_W)\), cf. (2.5), and then in Theorem 2.2 establish the equivalence of the length \( \ell_W \) and the \( E \) functional, the latter defined as \( E_0 \) for \( \delta = 0 \) in (2.3). Next, we show the existence of a curve minimizing the length \( \ell_W \) between any two points in \( \mathbb{R}^N \), such a curve will be referred to as minimizing geodesic, cf. Theorem 2.3.

In §2.3 we exhibit conditions under which minimizers of \( E \) yield minimizers of \( H \), hence solutions to (2.1), cf. Theorem 2.1. This naturally requires further regularity assumptions on \( W \) beyond continuity, see (A3), so as to make (2.1) meaningful; plus a subquadratic growth of the potential.
around the wells, (A4). The latter condition will be used to prove ad-hoc properties of the parameter of energy-equipartition, so that the program in §2.3 can be carried out, cf. Lemma 2.4.

When there are three or more potential wells, then making this logical bridge between $E$ minimizers and $H$ minimizers requires an additional assumption, namely that the minimizing geodesic joining two wells by solving (2.2) does not pass through any other wells on its way. This is precisely the obstruction to existence of heteroclinic connections that the authors of [3] first revealed for certain planar systems and which was also examined in detail in [8,9] when the potential takes the form $W(z) = |f(z)|^2$ where $f$ is holomorphic. Here we establish this necessary condition for non-existence for very general $W$, any $m \geq 3$, and $N$ arbitrary.

The statement of the main result of Chapter 2 is the following

**Theorem 2.1.** Given $m, N \geq 2$ suppose $W : \mathbb{R}^N \to [0, \infty)$ is a potential satisfying (A1) through (A4). Then any pair of wells $p_j, p_k$ can be joined with a curve $\gamma_*$ which minimizes the length $\ell_W$ amongst competitors having fixed endpoints $p_j$ and $p_k$. In addition,

(i) If $\gamma_*$ does not go through other wells at intermediate times, then there is of this curve $U(t) := \gamma_*(x(t))$ solving the heteroclinic connection problem (2.1).

(ii) If $\gamma_*$ visits at least three wells, say $\gamma_*(t_1) = p_{i_1}, \gamma_*(t_2) = p_{i_2}, \ldots, \gamma_*(t_J) = p_{i_J}$ (for $J \geq 3$) as the parameter increases, then every pair of consecutive wells $\gamma_*(t_k)$ and $\gamma_*(t_{k+1})$ can be joined with a heteroclinic connection, obtained as a reparametrization of the restriction $\gamma_*|_{(t_k, t_{k+1})}$.

For a precise statement of the result, see Theorem 2.3 and Theorem 2.4.

### 2.2 Existence of minimizing geodesics

In this section we establish the existence of curves solving the problem (2.2). Our approach leads us into the realm of length spaces.
2.2.1 Geometric framework

For $N \geq 2$, we let $W : \mathbb{R}^N \rightarrow [0, \infty)$ be any continuous function satisfying the following

(A1) The zero set $\mathcal{Z}$ of $W$ consists of $m$ distinct points $p_1, \ldots p_m$ so that $W(p_1) = \ldots = W(p_m) = 0$, and $W > 0$ elsewhere.

(A2) $\lim \inf_{|p| \to \infty} W(p) > 0$.

In order the make the notation somewhat simpler, we will work in this section with $F := \sqrt{W}$ rather than $W$. Throughout, we will write $B_r(p) := \{x \in \mathbb{R}^N : |x - p| < r\}$ for the Euclidean ball centered at $p \in \mathbb{R}^N$ with radius $r > 0$. Given any continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^N$ we denote the set of times at which the curve runs into the zeros of $F$ by

$$T^W_\gamma := \{t \in [0, 1] : \gamma(t) \in \mathcal{Z}\}.$$

A central role in our analysis will be played by the set of continuous curves defined on $[0, 1]$ whose restrictions to connected sub-arcs that have no intersection with the zeros of $F$ are locally Lipschitz continuous in $\mathbb{R}^N$, endowed with the standard Euclidean metric $| \cdot |$. We denote this class of curves by

$$\text{Lip}_W([0, 1]; \mathbb{R}^N) := \{\gamma \in C([0, 1]; \mathbb{R}^N) : \gamma \in \text{Lip}_{loc}(([0, 1] \setminus T^W_\gamma); \mathbb{R}^N)\}.$$

We remark that in the special case where $\gamma \in \text{Lip}_W([0, 1]; \mathbb{R}^N)$ is such that $T^W_\gamma = \emptyset$, then if fact $\gamma$ is a Lipschitz continuous curve. Also, for any two distinct points $p, q \in \mathbb{R}^N$, we denote

$$\text{Lip}_W(p, q) := \{\gamma \in \text{Lip}_W([0, 1]; \mathbb{R}^N) : \gamma(0) = p, \gamma(1) = q\}.$$

We consider now the functional $E : \text{Lip}_W([0, 1]; \mathbb{R}^N) \rightarrow \mathbb{R}$ that is used to define a notion of length of curves in $\text{Lip}_W([0, 1]; \mathbb{R}^N)$, using a metric conformal to the standard Euclidean one with $F$ as a degenerate conformal factor:

$$E(\gamma) := \int_0^1 F(\gamma(t)) |\gamma'(t)| \, dt.$$
**Definition 2.1.** Let us introduce a metric $d_W$ on $\mathbb{R}^N$ induced by the functional $E$ by letting

$$d_W(p,q) := \inf_{\gamma \in \text{Lip}_W(p,q)} E(\gamma) \quad \text{for any } p,q \in \mathbb{R}^N.$$  

(2.4)

This metric gives rise to a natural length structure associated to it, by means of

$$\ell_W(\gamma) := \sup_{\{t_j\}_{j=1}^N \in \Psi([0,1])} \sum_{j=1}^N d_W(\gamma(t_j), \gamma(t_{j+1})),$$

(2.5)

where $\Psi([0,1])$ is the set of finite partitions of $[0,1]$. The value $\ell_W(\gamma)$ will be called the length of a curve $\gamma$, and we will say that a curve $\gamma$ is $E$-rectifiable when it has finite length $\ell_W(\gamma) < \infty$.

Despite the degeneracy of $F$, it is easy to check that $d_W$ satisfies the properties of a metric on $\mathbb{R}^N$. It is worth mentioning that $d_W$ so defined makes $(\mathbb{R}^N, d_W)$ into a length space, in the sense that for the metric space $(\mathbb{R}^N, d_W)$, the value of $d_W(p,q)$ is equal to the infimum of the length of admissible curves joining $p$ and $q$, see [19, pp. 32].

Before proceeding, we make note of the easy inequality

$$d_W(p,q) \leq \ell_W(\gamma) \quad \text{for all } p,q \in \mathbb{R}^N \text{ and all curves } \gamma \in \text{Lip}_W(p,q),$$

that follows immediately from (2.5) by choosing the partition $P = \{0,1\}$ of $[0,1]$.

### 2.2.2 Equivalence of $E$ and $\ell_W$ for admissible curves

Our first goal is to prove that the values of $E(\gamma)$ and $\ell_W(\gamma)$ match for $\gamma$ in the class of admissible curves in consideration. To this end, we begin with a standard lower semi-continuity property of the length functional in general length spaces, see e.g. [19]. For the sake of completeness, however, the proof is included.

**Lemma 2.1.** Let $\{\gamma_n\}$ be a sequence of curves from $[0,1]$ to $\mathbb{R}^N$, converging uniformly to an $E$-rectifiable curve $\gamma_0$ in the $d$ metric. Then

$$\liminf_{n \to \infty} \ell_W(\gamma_n) \geq \ell_W(\gamma_0).$$
Proof of Lemma 2.1. Consider an $E$-rectifiable curve $\gamma_0 : [0, 1] \to \mathbb{R}^N$ and suppose $\gamma_n \Rightarrow \gamma_0$. Let \( \{t_j\}_{j=0}^J \) be any partition of $[0, 1]$. The uniform convergence yields for any $\varepsilon > 0$, $\exists n_0(\varepsilon)$ so that for $n \geq n_0(\varepsilon)$

$$\sup_{t \in [0, 1]} d_W(\gamma_n(t), \gamma_0(t)) < \frac{\varepsilon}{2J}.$$ 

It follows that for $n \geq n_0(\varepsilon)$, and all $0 \leq j \leq J$:

$$d_W(\gamma_0(t_j), \gamma_0(t_{j+1})) \leq d_W(\gamma_0(t_j), \gamma_n(t_j)) + d_W(\gamma_n(t_j), \gamma_n(t_{j+1})) + d_W(\gamma_n(t_{j+1}), \gamma_0(t_{j+1}))$$

$$\leq \frac{\varepsilon}{2J} + d_W(\gamma_n(t_j), \gamma_n(t_{j+1})) + \frac{\varepsilon}{2J}.$$ 

Adding these inequalities over $j \in \{0, \ldots, J\}$ we deduce

$$\sum_{j=0}^J d_W(\gamma_0(t_j), \gamma_0(t_{j+1})) \leq \ell_W(\gamma_0) + \varepsilon,$$

from which it follows $\sum_{j=0}^J d_W(\gamma_0(t_j), \gamma_0(t_{j+1})) \leq \liminf_{n \to \infty} \ell_W(\gamma_n) + \varepsilon$. Taking the supremum over all partitions of $[0, 1]$ we conclude that $\ell_W(\gamma_0) \leq \liminf_{n \to \infty} \ell_W(\gamma_n) + \varepsilon$, with $\varepsilon > 0$ arbitrary. \(\square\)

Next, we establish a lower-semi-continuity property of the functional $E$.

Lemma 2.2. Consider $\{\gamma_j\}$, $\gamma_0 \in \text{Lip}_W([0, 1]; \mathbb{R}^N)$. Then $E : \text{Lip}_W([0, 1]; \mathbb{R}^N) \to \mathbb{R}$ is a lower semi-continuous functional with respect to uniform convergence in the Euclidean metric:

$$\liminf_{j \to \infty} E(\gamma_j) \geq E(\gamma_0) \quad \text{whenever} \quad \sup_{t \in [0, 1]} |\gamma_j - \gamma_0| \to 0 \quad \text{as} \quad j \to \infty.$$ 

Proof of Lemma 2.2. With no loss of generality we assume $\liminf_{j \to \infty} E(\gamma_j) < \infty$. Fixing $\varepsilon > 0$, let us consider the punctured plane $\Omega_{2\varepsilon} := \mathbb{R}^N \setminus (\bigcup_{i=1}^m \{x : |x - p_i| < 2\varepsilon\})$, and in addition we introduce the set $T_{\varepsilon} = \{t \in [0, 1] : \gamma_0(t) \in \Omega_{2\varepsilon}\}$ with the possibility that $[0, 1] \setminus T_{\varepsilon}$ could be empty if $\gamma_0$ avoids $3$. The uniform convergence $\gamma_j \Rightarrow \gamma_0$ yields the existence of a value $\varepsilon_0(\varepsilon)$ such that

$$\bigcup_{J \geq J_0(\varepsilon)} \{\gamma_j(t) : t \in T_{\varepsilon}\} \subset \Omega_{\varepsilon}. \quad \text{Next, we decompose}$$

$$\int_{T_{\varepsilon}} F(\gamma_j) |\gamma_j'| \, dt = \int_{T_{\varepsilon}} (F(\gamma_j) - F(\gamma_0)) |\gamma_j'| \, dt + \int_{T_{\varepsilon}} F(\gamma_0) |\gamma_j'| \, dt. \quad (2.6)$$
The key observation is that \( \{ \gamma_j \} \) restricted to \( T_\varepsilon \) has bounded Euclidean arc-length:

\[
\liminf_{j \to \infty} \int_{T_\varepsilon} |\gamma'_j| \; dt \leq \frac{1}{c_\varepsilon} \liminf_{j \to \infty} \int_{T_\varepsilon} F(\gamma_j)|\gamma'_j| \; dt \leq \frac{1}{c_\varepsilon} \liminf_{j \to \infty} E(\gamma_j) \equiv C_\varepsilon < \infty.
\]

for \( c_\varepsilon = \min_{p \in \Omega_\varepsilon} F(p) > 0 \). In particular, upon the extraction of a subsequence, we may assume \( \| \gamma_j' \|_{L^1(T_\varepsilon)} \leq C_\varepsilon \) for all \( j \geq j_0(\varepsilon) \). By virtue of the uniform continuity of \( F \) on the compact set \( \{ p : \inf_{\varepsilon \in [0,1]} |p - \gamma_0(t)| \leq \delta \} \) for some \( \delta = \delta(\varepsilon) \), together with the uniform convergence \( \gamma_j \rightharpoonup \gamma_0 \) in the Euclidean metric, we get the bound \( \max_{t \in [0,1]} |F(\gamma_j) - F(\gamma_0)| < \varepsilon/C_\varepsilon \) for all \( j \geq j_1(\varepsilon) \).

Whence, choosing \( j \geq \max\{ j_0(\varepsilon), j_1(\varepsilon) \} \) it follows that

\[
\limsup_{j \to \infty} \left| \int_{T_\varepsilon} (F(\gamma_j) - F(\gamma_0))|\gamma'_j| \; dt \right| \leq \varepsilon. \tag{2.7}
\]

For the second term in (2.6), note that \( t \mapsto F(\gamma_0(t)) \) is a continuous positive function defined on \( T_\varepsilon \), thus we can use the following characterization of bounded variation for an \( L^1 \)-function:

\[
\int_{T_\varepsilon} F(\gamma_0)|\gamma'_j| \; dt = \sup \left\{ \int_{T_\varepsilon} \gamma_j(t) \cdot g'(t) \; dt : g \in C^1_c(T_\varepsilon; \mathbb{R}^N), |g| \leq F(\gamma_0) \text{ on } T_\varepsilon \right\}.
\]

Fix now such a vector field \( g \). The uniform convergence \( |\gamma_j - \gamma_0| \rightharpoonup 0 \) implies

\[
\liminf_{j \to \infty} \int_{T_\varepsilon} F(\gamma_0)|\gamma'_j| \; dt \geq \liminf_{j \to \infty} \int_{T_\varepsilon} \gamma_j(t) \cdot g'(t) \; dt = \int_{T_\varepsilon} \gamma_0(t) \cdot g'(t) \; dt,
\]

so taking the supremum over \( g \in C^1_c(T_\varepsilon; \mathbb{R}^N) \) with \( |g(t)| \leq F(\gamma_0(t)) \) on \( T_\varepsilon \) we arrive at

\[
\liminf_{j \to \infty} \int_{T_\varepsilon} F(\gamma_0)|\gamma'_j| \; dt \geq \int_{T_\varepsilon} F(\gamma_0)|\gamma'_0| \; dt. \tag{2.8}
\]

Applying the estimates (2.7) and (2.8) to the identity (2.6), one derives

\[
\liminf_{j \to \infty} \int_{T_\varepsilon} F(\gamma_j)|\gamma'_j| \; dt \geq \int_{T_\varepsilon} F(\gamma_0)|\gamma'_0| \; dt - \varepsilon.
\]

Now, the continuity of \( \gamma_0 \) ensures the convergence of the characteristic functions \( \chi_{T_\varepsilon} \rightarrow \chi_{[0,1]} \equiv 1 \) for a.e. \( t \in [0,1] \), as \( \varepsilon \to 0^+ \). Then the monotone convergence theorem applied to the above inequality proves the desired conclusion

\[
\liminf_{j \to \infty} E(\gamma_j) \geq \limsup_{\varepsilon \to 0^+} \left( \int_{T_\varepsilon} F(\gamma_0)|\gamma'_0| \; dt - \varepsilon \right) = \int_{[0,1]} F(\gamma_0)|\gamma'_0| \; dt = E(\gamma_0).
\]

\( \square \)
The main goal of this section is to establish the existence of $E$-minimizing curves joining two given points in $\mathbb{R}^N$. Here we present a preliminary result on the existence of minimizers joining two nearby points, both far away from the zero set $\mathcal{Z}$ of $W$. There the metric $d_W$ is locally equivalent to the standard Euclidean metric. Consequently, the existence of $E$-minimizing curves joining nearby points will follow easily from an application of the direct method in the calculus of variations, where compactness is recovered from the non-degeneracy of the metric $d_W$ away from the wells.

**Lemma 2.3.** For every $\varepsilon > 0$ such that $\mathcal{Z} \subset B_{1/\varepsilon}(0)$ there exists a number $r_\varepsilon > 0$ such that for all $p, q \in B_{1/\varepsilon}(0) \setminus \bigcup_{j=1}^m B_{2\varepsilon}(p_j)$ satisfying $|p - q| < r_\varepsilon$, there exists an $E$-minimizing curve joining $p$ to $q$ that avoids an $\varepsilon$-neighborhood of $\mathcal{Z}$.

**Proof of Lemma 2.3.** Given $\varepsilon > 0$, we define the three positive numbers

$$M_\varepsilon := \max_{\{p : |p| \leq 1/\varepsilon\}} F(p), \quad m_\varepsilon := \min_{\bigcup_j \{p : \varepsilon \leq |p - p_j| \leq 2\varepsilon\}} F(p) \quad \text{and} \quad r_\varepsilon := \frac{\varepsilon m_\varepsilon}{M_\varepsilon}.$$ 

Now we take distinct points $p, q \in B_{1/\varepsilon}(0) \setminus \bigcup_{j=1}^m B_{2\varepsilon}(p_j)$ satisfying $|p - q| \leq r_\varepsilon$ and let $\{\gamma_k\} \subset \text{Lip}_W(p, q)$ denote a minimizing sequence in (2.4) so that $E(\gamma_k) \to d(p, q)$. Denoting by $\alpha_{\text{aff}}$ the line segment joining $p$ to $q$ we may assume $E(\gamma_k) \leq E(\alpha_{\text{aff}})$ so that we obtain the upper bound

$$E(\gamma_k) \leq E(\alpha_{\text{aff}}) = \int_0^1 F(\alpha_{\text{aff}}(s))|p - q|\, ds \leq M_\varepsilon r_\varepsilon.$$  

In light of (2.9), we now claim that for each $k$, $\gamma_k$ cannot pass within $\varepsilon$-Euclidean distance of the zero set of $F$. To see this, note that if it did pass within $\varepsilon$ of $p_j$ for some $j$, then we would have the lower bound

$$E(\gamma_k) \geq \int_{\{t : \varepsilon \leq |\gamma_k(t) - p_j| \leq 2\varepsilon\}} F(\gamma_k(t))|\gamma_k'(t)|\, dt \geq 2\varepsilon m_\varepsilon,$$

which is impossible given the definition of $r_\varepsilon$, see Figure 2.1 below.

Once we know each $\gamma_k$ avoids an $\varepsilon$ neighborhood of $\mathcal{Z}$, we can invoke the assumption (A2) to easily obtain the compactness we need, in that for all $k$ we have

$$\int_0^1 |\gamma_k'|\, dt \leq \left(\min_{\bigcup_j \{p : |p - p_j| \geq \varepsilon\}} F(p)\right)^{-1} \int_0^1 F(\gamma_k)|\gamma_k'|\, dt \leq \left(\min_{\bigcup_j \{p : |p - p_j| \geq \varepsilon\}} F(p)\right)^{-1} (d_W(p, q) + 1).$$
We can then reparametrize $\gamma_k$ by constant speed $C_k := \int_0^1 |\gamma_k'| \, dt$ with $C_k$ bounded independent of $k$. Hence, $\gamma_k : [0, 1] \to \mathbb{R}^N$ are equi-Lipschitz. Applying the Arzela-Ascoli theorem, we have that upon extraction of a subsequence, there exists a Lipschitz curve $\gamma_0$ such that $\gamma_{k_\ell} \Rightarrow \gamma_0$ with respect to the Euclidean metric. The lower-semi-continuity result Lemma 2.2 then shows that

$$d_W(p, q) = \liminf_{\ell \to \infty} E(\gamma_{k_\ell}) \geq E(\gamma_0),$$

and so $\gamma_0$ minimizes $E$. \hfill \Box

The key result of this subsection is the assertion below that $E(\gamma)$ and $\ell_W(\gamma)$ coincide:

**Theorem 2.2.** For any curve $\gamma \in \text{Lip}_W([0, 1]; \mathbb{R}^N)$ one has the equivalence

$$\ell_W(\gamma) = E(\gamma).$$

**Proof of Theorem 2.2.** In light of the additivity property enjoyed by both functionals $E$ and $\ell_W$ on concatenation of curves, given any $\gamma \in \text{Lip}_W([0, 1]; \mathbb{R}^N)$, it will suffice to establish this equivalence on any arc of $\gamma$ such that $\gamma$ avoids the set $Z$ except perhaps at one or both of its endpoints. This follows since clearly an arbitrary curve can be decomposed into a union of such arcs. Pursuing the worst case scenario, with a slight abuse of notation we will then simply assume that $\gamma$ is one such arc with endpoints lying on distinct points $p_{k_1}$ and $p_{k_2}$ of $Z$. 

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We first observe that for any partition \(\{s_j\}_{j=0}^J\) of \([0,1]\) one has using the definition of the metric \(d_W\) on each arc \(\gamma([s_j,s_{j+1}])\) that

\[
\sum_{j=0}^{J-1} d_W(\gamma(s_j), \gamma(s_{j+1})) = \sum_{j=0}^{J-1} \inf_{\beta \in \text{Lip}_W(\gamma(s_j),\gamma(s_{j+1}))} E(\beta) \leq \sum_{j=0}^{J-1} E(\gamma|_{[s_j,s_{j+1}]}) = E(\gamma).
\]

Taking the supremum over all partitions of \([0,1]\) we conclude that \(\ell_W(\gamma) \leq E(\gamma)\). We now turn to the task of proving the reverse inequality. We remark that in the case the curve satisfies \(\ell_W(\gamma) = \infty\), then it immediately follows \(E(\gamma) = \infty\) and so \(\ell_W(\gamma) = E(\gamma)\). Let us then assume that \(\ell_W(\gamma) < \infty\). We will argue in this case that \(E(\gamma)\) is finite and further \(E(\gamma) \leq \ell_W(\gamma)\). The strategy is to construct a sequence \(\{\gamma_n\} \subset \text{Lip}_W([0,1];\mathbb{R}^N)\) with \(n \to \infty\) that interpolates on a set of \(n\) points on \(\gamma\), where the interpolating pieces are chosen to be \(E\)-minimizing.

First choose \(\varepsilon > 0\) sufficiently small so that \(\gamma((0,1)) \subset B_{1/\varepsilon}(0)\). Then define \(0 < t_\varepsilon < \bar{t}_\varepsilon < 1\) through the conditions

\[
t_\varepsilon := \max\{t : |\gamma(t) - p_{k_1}| \leq 2\varepsilon\}, \quad \bar{t}_\varepsilon := \min\{t : |\gamma(t) - p_{k_2}| \leq 2\varepsilon\}.
\]

The claim is that \(t_\varepsilon \to 0\) and \(\bar{t}_\varepsilon \to 1\) as \(\varepsilon \to 0^+\). By contradiction let us suppose that there exists a sequence \(\{t_{\varepsilon_j}\}\) with \(t_{\varepsilon_j} \to t_* \in (0,1)\) as \(\varepsilon_j \to 0^+\). It follows that \(|\gamma(t_{\varepsilon_j}) - p_{k_1}| = \varepsilon_j\) for every \(j\) by definition of \(t_{\varepsilon_j}\). Taking the limit \(j \to \infty\) in the above equality we deduce by the continuity of \(\gamma\) that \(|\gamma(t_*) - p_{k_1}| = 0\) and so \(\gamma(t_*) = p_{k_1}\), but this contradicts the assumption that the set \(\mathfrak{3}\) is avoided by \(\gamma\) at intermediate times. The second statement of the claim can be argued similarly.

For all \(\varepsilon > 0\) sufficiently small, one has \(\gamma([t_\varepsilon, \bar{t}_\varepsilon]) \subset B_{1/\varepsilon}(0) \setminus \bigcup_{i=1}^m B_{2\varepsilon}(p_i)\). Hence, Lemma 2.3 yields the existence of \(r_\varepsilon > 0\) so that any choice of points \(p,q \in \gamma([t_\varepsilon, \bar{t}_\varepsilon])\) with \(|p-q| < r_\varepsilon\) can be joined with an \(E\)-minimizing curve.

For all \(n\) sufficiently large, we now label \(p_1 := \gamma(t_\varepsilon)\) and \(p_{n-1} := \gamma(\bar{t}_\varepsilon)\) and then pick a collection of equispaced points \(\{p_2, \ldots, p_{n-2}\} \subset \gamma([t_\varepsilon, \bar{t}_\varepsilon])\) so that \(|p_j - p_{k+1}| < r_\varepsilon\) for \(j = 1, \ldots, n-2\). Let us write \(p_k = \gamma(t_k)\) with increasing \(t\)-values. By virtue of Lemma 2.3 on every pair of consecutive points we find \(E\)-minimizing curves, that up to reparametrization we write
as \( \alpha_k \in \text{Lip}(\{t_k, t_{k+1}\}; \mathbb{R}^N) \) with \( \alpha_k(t_k) = p_k \) and \( \alpha_k(t_{k+1}) = p_{k+1} \) for \( j = 1, \ldots, n - 2 \). We define \( \gamma_n \in \text{Lip}_W([0, 1]; \mathbb{R}^N) \) as the concatenation of \( \alpha_{\text{aff}}^-, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{\text{aff}}^+ \), where \( \alpha_{\text{aff}}^- : [0, t_\varepsilon] \to \mathbb{R}^N, \alpha_{\text{aff}}^+ : [t_\varepsilon, 1] \to \mathbb{R}^N \) are parametrizations of the linear segments \([p_{k_1}, p_1]\) and \([p_{n-1}, p_{k_2}]\), respectively, see Figure 2.2 below. One readily checks that as \( n \to \infty \) we have \( |\gamma_n(t) - \gamma(t)| \to 0 \) uniformly for \( t \in [t_\varepsilon, t_\varepsilon + 2\varepsilon] \).

![Figure 2.2: Construction of the interpolating sequence \( \{\alpha_k\}_{k=1}^{n-2} \), in the case \( n = 6 \).](image)

On the other hand, since \( \{\alpha_k\}_{k=1}^{n-2} \) are \( E \)-minimizers among Lipschitz continuous competitors with fixed endpoints we get

\[
\sum_{k=1}^{n-2} E(\alpha_k) = \sum_{k=1}^{n-2} d_W(p_k, p_{k+1}) = \sum_{k=1}^{n-2} d_W(\gamma(t_k), \gamma(t_{k+1})) \leq \ell_W(\gamma|_{[t_\varepsilon, t_\varepsilon + 2\varepsilon]}).
\]

We conclude that for all large \( n \) one has

\[
E(\gamma_n) = E(\alpha_{\text{aff}}^-) + \sum_{k=1}^{n-2} E(\alpha_k) + E(\alpha_{\text{aff}}^+) \leq \ell_W(\gamma) + 4M\varepsilon,
\]

where \( M := \max\{F(p) : p \in \overline{B_{2\varepsilon}(p_{k_1}) \cup B_{2\varepsilon}(p_{k_2})}\} \), using the trivial bound on the two linear pieces:

\[
E(\alpha_{\text{aff}}^\pm) \leq 2M\varepsilon.
\]

(We note that in fact \( M = M_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) but we won’t need this.) Hence,

\[
\limsup_{n \to \infty} E(\gamma_n) \leq \ell_W(\gamma) + 4M\varepsilon. \tag{2.11}
\]
Applying the estimates (2.10) and (2.11), we conclude that

\[ E(\gamma|_{[L, \bar{L}]}) \leq \ell_W(\gamma) + 4M\varepsilon. \]

Finally, by the claim above \( \ell_\varepsilon \to 0 \) and \( \bar{L}_\varepsilon \to 1 \) as \( \varepsilon \to 0 \), so one has \( E(\gamma|_{[L, \bar{L}]}) \to E(\gamma) \) as \( \varepsilon \to 0 \). Consequently,

\[ E(\gamma) \leq \ell_W(\gamma). \]

Thus, \( E(\gamma) = \ell_W(\gamma) \) for every \( \gamma \in \text{Lip}_W([0, 1]; \mathbb{R}^N) \).

\[ \square \]

### 2.2.3 Proof of existence of minimizing geodesics

Now we turn to the existence of \( E \)-minimizing curves joining any two points in \( \mathbb{R}^N \), including the case of joining two wells of the potential, where the conformal factor \( F := \sqrt{W} \) of \( d_W \) degenerates.

**Theorem 2.3.** *Given two distinct points \( p, q \in \mathbb{R}^N \), there exists a curve \( \gamma_* \in \text{Lip}_W(p, q) \) satisfying*

\[ \ell_W(\gamma_*) = \inf_{\gamma \in \text{Lip}_W(p, q)} \ell_W(\gamma) = d_W(p, q) = E(\gamma_*). \]  

(2.12)

We will refer to such a \( \gamma_* \) as a *minimizing geodesic between \( p \) and \( q \).* The proof is an application of the direct method in the context of length spaces, and is an adaptation of a proof of the Hopf-Rinow theorem for length spaces, see e.g. [19, pp. 35].

**Proof of Theorem 2.3.** The proof consists of demonstrating the existence of a curve in \( \text{Lip}_W(p, q) \) that minimizes the length functional \( \ell_W \). That such a minimizer also is a minimizer of \( E \) then follows immediately from Theorem 2.2. We will break the proof into two steps.

**Step 1.** We first exhibit a curve \( \gamma_* \) with minimal length having endpoints \( p \) and \( q \). By definition of the metric, there is a minimizing sequence \( \{ \gamma_j \} \subset \text{Lip}_W(p, q) \) with \( E(\gamma_j) \to d_W(p, q) \) as \( j \to \infty \). Then by Theorem 2.2, \( \lim_{j \to \infty} \ell_W(\gamma_j) = d_W(p, q) \) as well. With no loss of generality we assume \( F(\gamma_j(t)) > 0 \) for every \( t \in (0, 1) \) and all \( j \), since if \( F(\gamma_{j_0}(t_0)) = 0 \) for some \( t_0 \in (0, 1) \), then we can locally modify the curve \( \gamma_{j_0} \) around \( t_0 \) so that it satisfies the above property, with
We now claim that appropriately parametrized, the sequence \( \{\gamma_j\} \) is equi-Lipschitz in \((\mathbb{R}^N, d_W)\). For each \( j \):

- First we reparametrize \( \gamma_j \) using **degenerate arc-length**
  \[
  F(\gamma_j(l))|\gamma_j'(l)| = 1 \quad \text{for} \quad 0 \leq l \leq E(\gamma_j).
  \]

- Then we normalize the time-scale using the energy, by introducing the parameter
  \[
  \tilde{l} \equiv \frac{l}{E(\gamma_j)} \quad \text{for} \quad 0 \leq \tilde{l} \leq 1.
  \]

Working with this parameter, we write

\[
\tilde{\gamma}_j : [0, 1] \to \mathbb{R}^N \quad \text{with} \quad \tilde{\gamma}_j(\tilde{l}) \equiv \gamma_j(l).
\] (2.13)

Fixing any \( 0 \leq \tilde{l}_1 \leq \tilde{l}_2 \leq 1 \), for any given \( \delta > 0 \) it holds that \( E(\gamma_j) \leq d_W(p, q) + \delta \), provided \( j \geq j_0(\delta) \) large enough. Then we have the lower estimate

\[
\tilde{l}_2 - \tilde{l}_1 = \frac{1}{E(\gamma_j)} (l_2 - l_1) = \frac{1}{E(\gamma_j)} \int_{l_1}^{l_2} F(\gamma_j)|\gamma_j'| \, dl \\
\quad \geq \frac{1}{(d_W(p, q) + \delta)} E(\gamma_j|_{l_1, l_2}) \geq \frac{1}{(d_W(p, q) + \delta)} d_W(\gamma_j(l_1), \gamma_j(l_2)).
\]

In light of (2.13) we conclude for all \( j \geq j_0(\delta) \)

\[
d_W(\tilde{\gamma}_j(\tilde{l}_1), \tilde{\gamma}_j(\tilde{l}_2)) \leq (d_W(p, q) + \delta)|\tilde{l}_1 - \tilde{l}_2|,
\]

which shows that the family \( \{\tilde{\gamma}_j\} \) is equi-Lipschitz with respect to the \( d_W \) metric. In addition, this family is uniformly bounded in the \( d \) metric, since for any \( \delta \) small one has

\[
\sup_{\tilde{l} \in [0,1]} d_W(p, \tilde{\gamma}_j(\tilde{l})) = \sup_{\tilde{l} \in [0,1]} d_W(\tilde{\gamma}_j(0), \tilde{\gamma}_j(\tilde{l})) \leq d_W(p, q) + 1,
\]

which implies \( \cup_{j \geq j_0(\delta)} \tilde{\gamma}_j([0, 1]) \subset \{ z \in \mathbb{R}^N : d_W(p, z) \leq d_W(p, q) + 1 \} \). The Arzela-Ascoli theorem then ensures the existence of a subsequence \( \{\tilde{\gamma}_{j_n}\} \) converging \( d_W \)-uniformly to some curve \( \gamma_* \) that is Lipschitz in the \( d_W \)-metric. In particular, \( d_W(p, q) + \delta \) is a bound for the Lipschitz constant of \( \gamma_* \), but since \( \delta > 0 \) is arbitrary we arrive to the same conclusion with the sharpest bound \( d_W(p, q) \).
Now we check that $\gamma_*\equiv \gamma_*$ is a length-minimizing curve between $p$ and $q$. Recall that $\{\tilde{\gamma}_{j_n}\}$ is a minimizing sequence with $\tilde{\gamma}_{j_n}\Rightarrow \gamma_*$ in the $d_W$-metric. Then Lemma 2.1 and Theorem 2.2 apply to yield

$$d_W(p, q) \leq \ell_W(\gamma_*) \leq \liminf_{n\to\infty} \ell_W(\tilde{\gamma}_{j_n}) = \liminf_{n\to\infty} E(\tilde{\gamma}_{j_n}) = d_W(p, q).$$

Since by Theorem 2.2 we have $\ell_W(\gamma_*) = E(\gamma_*)$ this will complete the proof of (2.12) once we show that $\gamma_*$ lies in $\text{Lip}_W(p, q)$, namely that its restriction to any sub-arc that avoids $Z$ is Lipschitz continuous with respect to the Euclidean metric.

**Step 2.** To this end, consider the arc $\gamma_*([a, b])$ for any $(a, b) \subset \subset [0, 1]$ such that $\gamma_*([a, b]) \cap Z = \emptyset$. With an eye towards applying Lemma 2.3, we now fix $\varepsilon$ sufficiently small so that $\gamma_*([a, b]) \subset B_{1/\varepsilon}(0) \setminus \bigcup_{j=1}^{m} B_{2\varepsilon}(p_j)$. Then for any interval $[a_1, b_1] \subset [a, b]$, we consider a partition $a_1 = t_0 < t_1 < t_2 < \ldots < t_n = b_1$ such that for each $j$ we have $|\gamma(t_{j+1}) - \gamma(t_j)| < r_\varepsilon$, with $r_\varepsilon$ taken from that lemma. We get a collection $\{\alpha_{j_{n}}\}_{j_{n}=1}^{n}$ of $E$-minimizing curves joining the endpoints $\gamma_*(t_j)$ to $\gamma_*(t_{j+1})$ for $j = 0, 1, \ldots, n - 1$, and we know that they all avoid an $\varepsilon$-neighborhood of the wells.

Thus, for each $j$ we have

$$d_W(\gamma_*(t_j), \gamma_*(t_{j+1})) = E(\alpha_{j}) = \int_{t_j}^{t_{j+1}} F(\alpha_{j})|\alpha_{j}'|$$

$$\geq \left( \min_{\bigcup_{j} \{p: |p-p_j| \geq \varepsilon \}} F(p) \right) \int_{t_j}^{t_{j+1}} |\alpha_{j}'|$$

$$\geq \left( \min_{\bigcup_{j} \{p: |p-p_j| \geq \varepsilon \}} F(p) \right) |\gamma_*(t_{j+1}) - \gamma_*(t_j)|.$$}

But since $d_W(\gamma_*(t_j), \gamma_*(t_{j+1})) \leq L(t_{j+1} - t_j)$ for each $j$, where $L$ denotes the Lipschitz constant of $\gamma_*$ with respect to the $d_W$ metric, we can sum over $j$ to find that

$$|\gamma_*(b_1) - \gamma_*(a_1)| \leq \left( \min_{\bigcup_{j} \{p: |p-p_j| \geq \varepsilon \}} F(p) \right)^{-1} L(b_1 - a_1),$$

and so $\gamma_* \in \text{Lip}_W([0, 1]; \mathbb{R}^N)$.

In passing, we mention some basic properties of minimizing geodesics.

**Proposition 2.1.** For any two distinct points $p, q \in \mathbb{R}^N$, consider a length-minimizing curve $\gamma_* \in \text{Lip}_W(p, q)$. Then,
(i) The restriction of $\gamma_*$ to any arc is length-minimizing: For any $[a, b] \subset [0, 1],$

$$\ell_W(\gamma_*|_{[a,b]}) = d_W(\gamma_*(a), \gamma_*(b)).$$

(ii) The length of $\gamma_*$ is achieved by computing over any finite partition $\Psi_0$ of $[0, 1],$

$$\ell_W(\gamma_*) = \sum_{\{t_i\} \in \Psi_0} d_W(\gamma_*(t_i), \gamma_*(t_{i+1})).$$

Proof of Proposition 2.1. For the first point, if $\gamma_*$ were not length-minimizing in some subinterval $[a_0, b_0]$ then $d_W(\gamma_*(a_0), \gamma_*(b_0)) < \ell_W(\gamma_*|_{[a_0, b_0]})$, but this directly contradicts the length-minimality of the entire curve. Indeed, using the triangle inequality, and the additivity of the length functional on concatenations of curves one has

$$d_W(\gamma_*(0), \gamma_*(1)) \leq d_W(\gamma_*(0), \gamma_*(a_0)) + d_W(\gamma_*(a_0), \gamma_*(b_0)) + d_W(\gamma_*(b_0), \gamma_*(1))$$

$$< \ell_W(\gamma_*|_{[0,a_0]}) + \ell_W(\gamma_*|_{[a_0,b_0]}) + \ell_W(\gamma_*|_{[b_0,1]})$$

$$= \ell_W(\gamma_*).$$

This proves the length-minimality of any arc $\gamma_*([a,b])$ for every $[a, b] \subset [0, 1].$

For the second point, given any partition $\Psi_0 = \{t_i\}_{i=0}^{I+1}$ of $[0, 1]$, we use the length-minimality result just proved on every arc $\gamma_*([t_i, t_{i+1}])$. We deduce

$$\ell_W(\gamma_*) = \sum_{i=0}^{I} \ell_W(\gamma_*|_{[t_i,t_{i+1}]}) = \sum_{i=0}^{I} d_W(\gamma_*(t_i), \gamma_*(t_{i+1})), $$

in light of the $\ell_W$-additivity on concatenations of curves. \qed

Finally we address the regularity of minimizing geodesics under further smoothness assumptions on $F$ away from $\mathcal{Z}$, or equivalently, on $W$.

Proposition 2.2. In addition to assumptions (A1) and (A2), assume that $F \in C^{1,\alpha}_\text{loc}(\mathbb{R}^N \setminus \mathcal{Z})$ with $\alpha \in (0, 1)$. Then for distinct points $p_j, p_k \in \mathcal{Z}$, a minimizing geodesic $\gamma_*$ joining $p_j$ to $p_k$ admits a $C^{2,\alpha}$-parametrization along any connected arc that avoids $\mathcal{Z}$.

Remark 2.1. If we further assume that $F \in C^{k+1,\alpha}_\text{loc}(\mathbb{R}^N \setminus \mathcal{Z})$ for some $k \geq 1$ and $\alpha \in (0, 1)$, then the same conclusion in Proposition 2.2 holds with $C^{k+2,\alpha}$-parametrizations.
Proof of Proposition 2.2. Fix any interval \([a, b] \subset (0, 1)\) such that \(\gamma_*([a, b]) \cap \mathcal{Z} = \emptyset\). The regularity established in Step 2 of Theorem 2.3 yields that this arc of \(\gamma_*\) is Lipschitz continuous with respect to the Euclidean metric. Hence, by the Rademacher’s theorem \(\gamma'_*\) exists a.e. in \([a, b]\). We note that if \(\gamma'_*(t) = 0\) for a.e. \(t\) in some interval \(I \subset [a, b]\), then we can simply reparametrize this curve by removing \(I\) and so with no loss of generality we may assume that \(|\gamma'_*(t)| \neq 0\) for a.e. \(t \in [a, b]\).

Now we can reparametrize \(\gamma_*\) restricted to \([a, b]\) by a multiple of Euclidean arc-length, choosing \(s = s(t) := l_*^{-1} \int_a^t |\gamma'_*(\tau)|d\tau\) with \(l_* := \int_a^b |\gamma'_*(t)|dt\) so that in the new parametrization of this arc one has \(\gamma_* : [0, 1] \to \mathbb{R}^N\) satisfying

\[
\left| \frac{d\gamma_*}{ds}(s) \right| = l_* \quad \text{for a.e. } s \in (0, 1).
\]

Taking any curve \(\gamma \in \text{Lip}([0, 1]; \mathbb{R}^N)\) satisfying \(\gamma(0) = \gamma(1) = 0\), we find that the arc of the minimizing geodesic \(\gamma_*\) under consideration satisfies the criticality condition

\[
0 = \delta E(\gamma_*)[\gamma] = \frac{d}{d\lambda} \bigg|_{\lambda = 0} \int_0^1 F(\gamma_* + \lambda \gamma) \left( \frac{d\gamma_*}{ds} + \lambda \frac{d\gamma}{ds} \right) \bigg| = \int_0^1 \left\{ \frac{1}{l_*} F(\gamma_*) \frac{d\gamma_*}{ds} \cdot \frac{d\gamma}{ds} + l_* \nabla_p F(\gamma_*) \cdot \gamma \right\} ds,
\]

and so is a weak solution of the Euler-Lagrange equation

\[
\frac{d}{ds} (F(\gamma_*) \frac{d\gamma_*}{ds} = l_*^2 \nabla_p F(\gamma_*) \quad \text{in } (0, 1).
\]

Now we may apply standard regularity theory. Since \(F(\gamma_*)\) is a positive, Lipschitz continuous function on \((0, 1)\) and the right-hand side lies in \(L^2((0, 1); \mathbb{R}^n)\), we conclude that \(\gamma_* \in H^2_{loc}((0, 1); \mathbb{R}^N)\) (see [29, Thm. 8.8]). Hence, by Sobolev embedding we have \(\gamma_* \in C^{1, \alpha}_{loc}((0, 1); \mathbb{R}^N)\). But then, we can view \(\gamma_*\) as a weak solution of the following differential equation

\[
\frac{d^2}{ds^2} (F(\gamma_*) \frac{d\gamma_*}{ds}) = -\left( \nabla_p F(\gamma_*) \cdot \frac{d\gamma_*}{ds} \right) \frac{d\gamma_*}{ds} + l_*^2 \nabla_p F(\gamma_*) =: G \quad \text{in } (0, 1),
\]

with \(G \in C^{0, \alpha}_{loc}((0, 1); \mathbb{R}^N)\). It then immediately follows that \(\gamma_* \in C^{2, \alpha}_{loc}((0, 1); \mathbb{R}^N)\).

In case \(F \in C^{k+1, \alpha}_{loc}(\mathbb{R}^N \setminus \mathcal{Z})\), a standard bootstrap argument allows one to deduce that \(\gamma_* \in C^{k+2, \alpha}_{loc}((0, 1); \mathbb{R}^N)\) for \(k \geq 2\). \(\square\)
2.3 Proof of the main Theorem 2.1

2.3.1 Existence through reparametrization of minimizing geodesics

In this section we establish the existence of heteroclinic connections between two wells of the multi-well potential $W = F^2$. Let us recall the assumptions on the potential:

(A1) The zero set $\mathcal{Z}$ of $W$ consists of $m$ distinct points $p_1, \ldots, p_m$ so that $W(p_1) = \ldots = W(p_m) = 0$, and $W > 0$ elsewhere.

(A2) $\lim inf_{|p| \to \infty} W(p) > 0$.

(A3) $W \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \mathcal{Z})$.

(A4) There exist $C, \delta > 0$ so that $W(p) \leq C|p - p_j|^2$ for every $p \in \bigcup_{j=1}^{m} B_\delta(p_j)$.

For the study of heteroclinic connections we re-introduce the functional $H$ defined on $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$, given by

$$H(U) := \int_{-\infty}^{+\infty} \frac{1}{2} |U'|^2 + W(U).$$

Following the scheme developed in [79], we study the connection problem by carrying out an analysis of the link between the geometric problem

$$\text{(GP)} \quad \inf_{\gamma \in \text{Lip}_W(p_j, p_k)} E(\gamma),$$

that was the focus of §2, and the variational problem

$$\text{(HP)} \quad \inf_{\substack{U \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N) \\\ U(x) \to p_j \text{ as } x \to -\infty \\\ U(x) \to p_k \text{ as } x \to +\infty}} H(U),$$

whose critical points yield heteroclinic connections between two distinct wells $p_j, p_k \in \mathcal{Z}$. We recall that the analysis in §2.2.3 yielded solutions to (GP), namely the minimizing geodesics of Theorem 2.3. We will argue that under an appropriate parametrization, these geodesics give rise to solutions to problem (HP) as well, though as we will see, the connected wells may differ from
\(
p_j \) and \( p_k \). Hence, in particular they will satisfy the associated Euler-Lagrange equation,

\[
U'' - \nabla_u W(U) = 0 \quad \text{in} \ (-\infty, +\infty),
\]

\[
U(-\infty) = p_j, \quad U(+\infty) = p_k,
\]

thus giving rise to a heteroclinic connection \( U \) between two elements \( p_j \) and \( p_k \) of \( \mathcal{Z} \).

Let us now restate in a more precise language the main result of Chapter 2, Theorem 2.1, which establishes the existence of an \( H \)-minimizing heteroclinic connection between two wells of \( W \), provided the trajectory of some minimizing geodesic solving the geometric problem (GP) joining these wells does not visit some other zero of \( W \) along the way.

**Theorem 2.4.** Given \( m,N \geq 2 \) suppose \( W : \mathbb{R}^N \to [0, \infty) \) is a potential satisfying (A1) through (A4). For distinct wells \( p_j,p_k \in \mathcal{Z} \) consider a minimizing geodesic \( \gamma_* \in \text{Lip}_W(p_j,p_k) \). Let us write

\[
0 = t_1 < t_2 < t_3 < \ldots < t_J = 1 \quad \text{(with} \ J \geq 2) \quad \text{for the times when} \ \gamma_*(t) \in \mathcal{Z}, \ \text{so that, in particular,} \ \gamma_*(t_1) = p_j \ \text{and} \ \gamma_*(t_J) = p_k. \ \text{Then for every} \ i \in \{1,\ldots,J-1\} \ \text{there exists an} \ H \text{-minimizing heteroclinic connection between the wells} \ \gamma_*(t_i) \ \text{and} \ \gamma_*(t_{i+1}).
\]

This theorem yields a sufficient condition, on the positioning of the wells of \( W \), for the existence of an \( H \)-minimizing heteroclinic connection between two given wells of a multi-well potential, for any number of wells \( m \geq 3 \) and any dimension \( N \geq 2 \). Furthermore, it provides a geometric description of such a heteroclinic connection. We have,

**Corollary 2.1.** If \( J = 2 \) in Theorem 2.4, that is, if a minimizing geodesic connecting two zeros \( p_j \) and \( p_k \) of \( W \) avoids any other zeros of \( W \) along the way, or equivalently, if the strict triangle inequality

\[
d_W(p_j,p_k) < d_W(p_j,p_l) + d_W(p_l,p_k) \quad \text{holds for all} \ \ p_l \in \mathcal{Z} \setminus \{p_j,p_k\}, \quad (2.14)
\]

then under an equipartition parametrization, this geodesic represents an \( H \)-minimizing connection between the two zeros.

The sufficiency of the strict inequality (2.14) had already been noted in [3] to yield existence of heteroclinic connections in the planar setting \( N = 2 \) (cf. (1.14) in [3]). Years later the authors
in [9] further established (2.14) as a necessary condition for the existence of \( \mathbb{R}^2 \)-valued heteroclinic connections between \( p_j \) and \( p_k \). See the discussion in §2.3.2 for details on the rigidity of the result in [9].

Another immediate consequence of Theorem 2.4 is the existence of a heteroclinic connection between the wells of a two-well potential, in arbitrary dimensions \( N \geq 2 \).

**Corollary 2.2 (Two-well case).** Assume the zero set \( \mathcal{Z} \) of \( W \) consists of exactly two points, \( p_1 \) and \( p_2 \). Then there exists an \( H \)-minimizing heteroclinic connection between these wells.

Before beginning the proof of the main Theorem 2.4, we remark on some properties of the so-called parametrization in *equipartition of energy* which plays a crucial role in our approach, (cf. [79, Lem. 1]):

**Lemma 2.4.** For any curve \( \gamma \in C([0,1]; \mathbb{R}^N) \cap C^1((0,1); \mathbb{R}^N) \) with non-vanishing derivative satisfying \( \gamma(0) = p_j \), \( \gamma(1) = p_k \) for \( p_j, p_k \in \mathcal{Z} \) and \( W(\gamma(t)) > 0 \) for all \( t \in (0,1) \), define the equipartition parameter \( x : (0,1) \to \mathbb{R} \) via

\[
x(t) := \int_{\frac{1}{2}}^{t} \frac{|\gamma'(u)|}{\sqrt{2W(\gamma(u))}} du.
\]

Then \( x : (0,1) \to \mathbb{R} \) is smooth, increasing and satisfies \( x((0,1)) = \mathbb{R} \). Furthermore, the curve \( U : \mathbb{R} \to \mathbb{R}^N \) given by \( U(x) := \gamma(t(x)) \) satisfies a pointwise equipartition of energy in the sense that

\[
\sqrt{2W(U)} \left| \frac{dU}{dx} \right| = \frac{1}{2} \left| \frac{dU}{dx} \right|^2 + W(U) \quad \text{for all } x \in \mathbb{R}.
\]

**Proof of Lemma 2.4.** We will write \( \gamma' = \frac{d\gamma}{dt} \). That \( x(t) \) is smooth and increasing is obvious. To establish that the range is all of \( \mathbb{R} \), we first reparametrize \( \{\gamma(t) : 0 < t < 1\} \) by Euclidean arc-length, denoted by \( s \), so that the equipartition parameter becomes

\[
x(s) = \frac{1}{\int_{s(1/2)}^{s} \frac{1}{\sqrt{2W(\gamma(t))}} d\tau} \quad \text{for } s(1/2) = \int_{0}^{1/2} |\gamma'(t)| dt.
\]

We will argue that \( \lim_{s \to 0^+} x(s) = -\infty \). The fact that \( \lim_{s \to s(1)^+} x(s) = +\infty \) will follow similarly. To this end, we fix \( \eta > 0 \) so that \( |\gamma(s) - p_j| < \delta \) for all \( s \in (0,\eta) \) where \( \delta \) is the value from assump-
tion (A4). In light of the arclength parametrization, we know that $|\gamma(s) - p_j| \leq s$. Consequently, for $s \in (0, \eta)$ one has
\[
W(\gamma(s)) \leq C|\gamma(s) - p_j|^2 \leq Cs^2.
\]
Then we find that
\[
\lim_{s \to 0^+} |x(s)| \geq \lim_{s \to 0^+} \int_s^\eta \frac{1}{\sqrt{2W(\gamma(\tau))}} \, d\tau \geq \lim_{s \to 0^+} \frac{1}{\sqrt{2C}} \int_s^\eta \frac{1}{\tau} \, d\tau = \infty.
\]
The third point is an easy consequence of the chain rule:
\[
\frac{1}{2} \frac{d^2}{dx^2} = \frac{1}{2} \frac{\gamma'}{dx} \cdot \frac{2W(U)}{\gamma^2} = W(U).
\]

**Proof of Theorem 2.4.** We consider first the case where $W(\gamma_*(t)) > 0$ for all $0 < t < 1$, so that $p_j$ and $p_k$ are the only zeros of $W$ traversed by the curve. By Proposition 2.2 we may take a $C^{2,\alpha}$-parametrization of $\gamma_*$, defined on $(0, 1)$, with non-vanishing derivative. Then applying Lemma 2.4, we reparametrize this curve using the equipartition parameter (2.15), yielding $U_*(x) := \gamma_*(t(x))$ with $U_* : (-\infty, +\infty) \to \mathbb{R}^N$. Since $E$ is invariant under reparametrization, the fact that $\gamma_*$ is a minimizing geodesic between $p_j$ and $p_k$, implies that $U_*$ solves the geometric problem (GP) as well, so that $E(U_*) = E(\gamma_*) = d_W(p_j, p_k)$. On the other hand, by virtue of the equipartition of energy in Lemma 2.4, we have $E(U_*) = H(U_*)$. Consequently, we deduce that for any $U \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ satisfying $U(-\infty) = p_j, U(+\infty) = p_k$
\[
H(U_*) = E(U_*) \leq E(U) \leq H(U).
\]
Hence, $U_*$ is a global minimizer of problem (HP). In view of the $C^{2,\alpha}$-regularity of $U_*$ guaranteed by Proposition 2.2, this curve is a classical solution to the Euler-Lagrange equation associated with $H$. We conclude that $U_*$ is a $H$-minimizing heteroclinic connection between $p_j$ and $p_k$.

To handle the case where $\gamma_*((0,1)] \cap \mathcal{Z} \neq \emptyset$, as previously remarked in Proposition 2.2 the geodesic $\gamma^*$ can only traverse $\mathcal{Z}$ finitely many times, so we write $0 = t_1 < t_2 < t_3 < \ldots < t_J = 1$.
with $J \geq 2$ for the times when $\gamma_s(t) \in \mathfrak{z}$. Fixing any $i \in \{1, \ldots, J - 1\}$ and restricting to the arc $\gamma_s((t_i, t_{i+1}))$ which does not intersect $\mathfrak{z}$, we apply the previous case to conclude that $U_s(x) := \gamma_s(t(x))$ is an $H$-minimizing heteroclinic connection between $\gamma_s(t_i), \gamma_s(t_{i+1}) \in \mathfrak{z}$, for $t \in (t_i, t_{i+1})$. \hfill \square 

### 2.3.2 Remarks on the obstruction to existence of heteroclinic orbits

As observed in [3, 8, 9] the presence of multiple wells may obstruct the existence of heteroclinic connections between two given wells $p_j$ and $p_k$ in $\mathfrak{z}$. In particular, utilizing complex variables techniques in the planar case $N = 2$ under the assumption that $W(z) = |f(z)|^2$ with $f$ holomorphic, the authors of [9] obtain various conditions, both necessary and sufficient, for existence of heteroclinic connections. For example, in case $W(z)$ takes the form

$$W(z) = \left| \prod_{j=1}^{3} a(z - z_j) \right|^2,$$

for some $a \in \mathbb{C}$, with $z_j \in \mathbb{C}$ taking the role of our $p_j$, their Proposition 4.5 states that a heteroclinic connection exists between, say, $z_1$ and $z_2$ if and only if the strict triangle inequality holds for the metric $d$:

$$d_W(z_1, z_2) < d_W(z_1, z_3) + d_W(z_3, z_2).$$

As a concrete example of the non-existence phenomenon, they consider the three-well potential $W_\varepsilon : \mathbb{C} \to \mathbb{R}$ given by $W_\varepsilon(z) = |(1 - z^2)(z - i\varepsilon)|^2$, for $\varepsilon \in \mathbb{R}$ and $z \in \mathbb{C}$, whose zero set is $\mathfrak{z} = \{-1, +1, i\varepsilon\}$. Their analysis proves that there exists a connection between $-1$ and $+1$ if and only if $|\varepsilon| > \sqrt{2\sqrt{3} - 3} =: \varepsilon_*$. In particular, when $|\varepsilon| \leq \varepsilon_*$ they establish the identity $d_W(-1, +1) = d_W(-1, i\varepsilon) + d_W(i\varepsilon, +1)$, leading to the conclusion that the minimizing geodesic between $-1$ and $1$ passes through the zero $i\varepsilon$, see the Figure 2.3 below.

This illustrates the obstruction to be avoided in applying our Corollary 2.1. Of course, in the generality in which our result holds, we only have the identity

$$d_W(p_j, p_k) = d_W(p_j, p_l) + d_W(p_l, p_k) \quad \text{for some } p_l \in \mathfrak{z},$$
as a necessary condition for non-existence of a heteroclinic connection between $p_j$ and $p_k$ in $Z$. Indeed, the striking rigidity of the Alikakos-Fusco result quoted above is surely related to the analyticity of $f$ in the assumption $W(z) = |f(z)|^2$, in addition to its being in the planar setting. For example, we see no reason why local minimizers or even saddle points of $E$ or of $H$ should not exist for general $W : \mathbb{R}^N \to [0, \infty)$ having three or more zeros, leading to connections even when the (globally) minimizing geodesic fails to provide a heteroclinic connection because it passes through a third well.

Though we do not present an explicit example of such an occurrence, we will conclude with an example of a non-minimizing heteroclinic connection that co-exists with multiple minimizing geodesic connections. Let us consider for $N = 3$ the potential $W : \mathbb{R}^3 \to \mathbb{R}$ given by

$$W(x, y, z) = x^2(1 - x^2)^2 + \left(y^2 - \frac{1}{2}(1 - x^2)^2\right)^2 + \left(z^2 - \frac{1}{2}(1 - x^2)^2\right)^2.$$ 

It can be readily checked that conditions (A1)-(A4) are satisfied by $W$. In particular we have that $m = 6$, and the collection of zeros of $W$ consists of

$$p_1 = (-1, 0, 0), \quad p_2 = (1, 0, 0),$$

$$p_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad p_4 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad p_5 = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad p_6 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

On the one hand, the existence of a heteroclinic connection between $p_1$ and $p_2$ can be argued by pursuing the ansatz $U_0(t) = (u(t), 0, 0)$ and then noting that the system of ODEs $U''_0 = \nabla W(U_0)$ reduces to a scalar differential equation

$$u'' = -2u(1 - u^2)(1 + u^2 + u^4) \quad \text{with} \quad u(\pm \infty) = \pm 1.$$
Existence of such a solution $u$ follows from an elementary phase plane analysis and the use of the pointwise equipartition relation $|U_0'|^2 = 2W(U_0)$. Hence, one gets the existence of a heteroclinic orbit $U_0 : \mathbb{R} \to \mathbb{R}^3$ joining $p_1$ to $p_2$ that follows the $x$-axis.

On the other hand, this line segment is not a minimizing geodesic between $p_1$ and $p_2$ since we can easily exhibit competitors with less $E$-values. For instance, consider

$$
\gamma_{\pm,\pm}(t) := \left( t, \frac{\pm .96}{\sqrt{2}}(1-t^2), \frac{\pm .96}{\sqrt{2}}(1-t^2) \right) \text{ for } t \in [-1, 1],
$$

which yields four curves joining $p_1$ to $p_2$. If we let $\gamma_0(t) = (t, 0, 0)$ be a parametrization of the line segment, we explicitly compute

$$
E(\gamma_{\pm,\pm}) = \int_{-1}^{1} \sqrt{W(\gamma_{\pm,\pm})} |\gamma_{\pm,\pm}'| dt
= \int_{-1}^{1} \sqrt{t^2(1-t^2)^2 + \frac{1}{2}(1-t^2)^4(1- (.96)^2)^2 [1 + 4t^2 - 4(1 - (.96)^2)t^2]} dt
\approx 0.74,
$$

while

$$
E(\gamma_0) = \int_{-1}^{1} \sqrt{W(\gamma_0)} |\gamma_0'| dt
= \int_{-1}^{1} \sqrt{t^2(1-t^2)^2 + (1-t^2)^4/2} dt
\approx 0.98.
$$

It then follows that $U_0$ is not a global minimizer of $H$ either. To see this, note that since, for example, $\gamma_{+,+} : [-1, 1] \to \mathbb{R}^3$ is a competing curve which does not run into any of the zeros $3 = \{p_1, \ldots, p_6\}$ of $W$ other than at the endpoints, we can reparametrize it by the equipartition parameter, cf. Lemma 2.4, to get a new curve $U_1 : (-\infty, \infty) \to \mathbb{R}^3$. Then the value of the two functionals $E$ and $H$ agree at this curve, so we conclude from the comparison of degenerate lengths above that

$$
H(U_1) = E(U_1) = E(\gamma_{+,+}) < E(\gamma_0) = E(U_0) \leq H(U_0).
$$

In light of these facts and invariance of the potential $W$ under the reflections $y \mapsto -y$, $z \mapsto -z$, there will exist multiple minimizing geodesics joining $p_1$ to $p_2$ in addition to the non-minimizing heteroclinic connection along the $x$-axis.
3.1 Introduction

In this section we go over, with greater level of detail than in Chapter 1, the existing result in [6] about the existence of vector-valued periodic solutions of the heteroclinic connection equation. Our main result of this chapter, Theorem 3.1, describes the relationship of these objects with a heteroclinic connection, in a sense to be made precise at the end of this section.

3.1.1 Periodic solutions to the heteroclinic connection system

Let us assume $W \in C^1(\mathbb{R}^N)$ is a non-negative potential with two wells $p_-$ and $p_+$ satisfying

\begin{equation}
\text{(W)} \quad \text{There exist closed, bounded sets } W^+, W^- \subset \mathbb{R}^N \text{ with } p_- \in \text{int}(W^-), \ p_+ \in \text{int}(W^+), \text{ and such that } \{W \leq c\} = W^- \cup W^+, \text{ while } W^- \cap W^+ = \emptyset.
\end{equation}

Henceforth we will write interchangeably

\begin{equation}
\{W \leq c\}^+ := W^+ \quad \text{and} \quad \{W \leq c\}^- := W^-,
\end{equation}

for such sets. Let us observe, by imposing some growth condition of $W$ around the wells, we get that $\{W \leq c\}^\pm = \{W \leq c\} \cap B_r(p_\pm)$ for some $r > 0$ sufficiently small and for small values of $c \approx 0$. We are interested in solutions to the system

\begin{equation}
q''(t) = \nabla W(q(t)) \quad \text{in } (-\infty, +\infty),
\end{equation}

(3.1)
whose orbit joins the two components of the sublevel set \( \{ W \leq c \}^- \) and \( \{ W \leq c \}^+ \),

\[
\inf_{t \in \mathbb{R}} \text{dist}(q(t), W^-_c) = \inf_{t \in \mathbb{R}} \text{dist}(q(t), W^+_c) = 0, \tag{3.2}
\]

where \( \text{dist}(x, A) := \inf \{|x - y| : y \in A\} \) is the Euclidean distance from \( x \) to the set \( A \subset \mathbb{R}^N \), and

in addition having prescribed pointwise mechanical energy at level \(-c\), namely,

\[
E_q := \frac{1}{2} |q'(t)|^2 - W(q(t)) = -c, \quad \text{for all } t \in \mathbb{R}. \tag{3.3}
\]

The existence of such orbits has been established by Montecchiari and Alessio in an unpublished manuscript [6]. Their main result shows that there exists \( q \in C^2(\mathbb{R}; \mathbb{R}^N) \cap L^\infty(\mathbb{R}; \mathbb{R}^N) \) solving (3.1) and satisfying (3.3); their approach is based on finding a minimizer of the following variational problem “at level \( c \)”,

\[
m_c = \inf \{ H^c(q) : q \in \mathcal{M}_c \}, \tag{BTPc}
\]

where \( H^c \) is a vector-version of the Modica-Mortola energy, renormalized to level \( c \),

\[
H^c(q) := \int_{-\infty}^{+\infty} \frac{1}{2} |q'(t)|^2 + (W(q(t)) - c) \, dt, \tag{3.4}
\]

and the admissible class is

\[
\mathcal{M}_c := \{ q \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) : \begin{array}{l}
(i) \quad W(q(t)) \geq c \text{ for any } t \in \mathbb{R}, \text{ and } \\
(ii) \quad \liminf_{t \to -\infty} \text{dist}(q(t), W^-_c) = \liminf_{t \to +\infty} \text{dist}(q(t), W^+_c) = 0 \}. \tag{3.5}
\]

Condition (i) constitutes an energy constraint, in that, the function \( W(q(t)) - c \) is non-negative over \( \mathbb{R} \) and so the functional \( H^c \) is well defined, and bounded from below on \( \mathcal{M}_c \). This variational scheme requires \( H^c \) to be coercive over \( \mathcal{M}_c \), meaning that the value of the infimum \( m_c \) as defined in (BTPc) does not increase when minimizing over a reduced admissible set with \( L^\infty \)-constraint. In other words, there exists \( R_0 > 0 \) so that

\[
m_c = \inf \{ H^c(q) : q \in \mathcal{M}_c, \| q \|_{L^\infty(\mathbb{R}; \mathbb{R}^N)} \leq R_0 \}. 
\]
This condition holds when, for example, one has a control of the behavior of the potential at infinity: \( \liminf_{|p| \to \infty} W(p) > 0 \).

Applying the direct method in the calculus of variations the authors obtain a minimizer \( q_c \in M_c \) of \( (BTPc) \). They further observe that \( q_c \) is a solution of (3.1) on any interval \( I \subset \mathbb{R} \) for which condition (i) is strictly satisfied, i.e., \( W(q_c(t)) > c \) for \( t \in I \). Indeed, by considering perturbations \( \psi \in C^1_c(I; \mathbb{R}^N) \) one readily sees that \( q_c + t\psi \in M_c \), so the criticality condition \( \delta H^c(q_c)[\psi] = 0 \) proves that \( q_c \) solves (3.1) on \( I \). This, in turn, implies that condition (ii) forces \( q_c \) to effectively connect \( W_c^- \) to \( W_c^+ \). Indeed, there must exist an interval \( I = (a, b) \subset \mathbb{R} \) (possibly unbounded) on which

\[
W(q_c(t)) > c \quad \text{for any } t \in (a, b), \quad \text{and} \quad \lim_{t \to a^+} \text{dist}(q_c(t), W_c^-) = \lim_{t \to b^-} \text{dist}(q_c(t), W_c^+) = 0,
\]

while (3.6) holds. In this case we refer to \( I \) as a connecting time interval for \( q_c \). Moreover, a further analysis shows that for a connecting interval one necessarily has

\[
H^c(q_c) = H^c_{(a,b)}(q_c),
\]

where we have employed the convention

\[
H_I^c(q) := \int_I \frac{1}{2}|q'(t)|^2 + (W(q(t)) - c) \, dt \quad \text{for } I \subset \mathbb{R}.
\]

Since the integrand in \( H^c(q) \) is always non-negative for \( q \in M_c \), we conclude from (3.7) that \( q_c(t) \equiv q_c(a) \) for all \( t \leq a \) and \( q_c(t) \equiv q_c(b) \) for all \( t \geq b \). It is therefore possible to make a unique choice of such connecting interval, that we will denote by \( I = (\alpha_c, \omega_c) \).

**Note.** We have intentionally adopted a cumbersome notation for \( q_c, \alpha_c \) and \( \omega_c \) because for the most part we will actually deal with corresponding the \( q_c, \alpha_c \) and \( \omega_c \), obtained through a renormalization procedure (3.19).

In particular, (3.7) combined with the energy constraint (3.3) yields the identity

\[
H^c(q_c) = 2 \int_{\alpha_c}^{\omega_c} (W(q_c) - c).
\]
In the event that \( c \) is a regular value of \( W \) (i.e. \( \{ W = c \} \subset \{ \nabla W \neq 0 \} \)) the authors are able to argue that these times \( \overline{\alpha}_c, \overline{\omega}_c \) are actually finite, and that \( q_c(\overline{\alpha}_c) \in \mathcal{W}_c^- \) and \( q_c(\overline{\omega}_c) \in \mathcal{W}_c^+ \). We say that these points are \textit{contact points} of the trajectory \( q_c \) with \( \mathcal{W}_c^\pm \), and that \( \overline{\alpha}_c, \overline{\omega}_c \) are \textit{contact times}.

Let us observe that if \( t \) is a contact time then \( W(q_c(t)) \leq c \), and so the energy condition (3.3) imposes that \( W(q_c(t)) = c \) and \( q_c'(t) = 0 \). Hence, \( t \) can be considered a \textit{turning time}, i.e., a time at which the orbit \( q_c \) is symmetric with respect to \( t \). This argument applies for any such contact time, including \( t = \overline{\alpha}_c \) and \( t = \overline{\omega}_c \). Thus a solution \( q \) to (3.1) satisfying (3.2) and (3.3) is built periodically by extending the minimizer of (BTPc) at the contact times \( \overline{\alpha}_c \) and \( \overline{\omega}_c \). More explicitly, if \( \Delta_c := \overline{\omega}_c - \overline{\alpha}_c \) denotes the half-period then the new orbit \( q \) is given by

\[
q(t) = \begin{cases} 
q_c(t - j\Delta_c) & \text{if } j \in 2\mathbb{Z}, \\
q_c((\overline{\alpha}_c + (j + 1)\Delta_c) - t + \overline{\alpha}_c) & \text{if } j \in (2\mathbb{Z} + 1),
\end{cases}
\]  

provided

\[
t \in [\overline{\alpha}_c + j\Delta_c, \overline{\alpha}_c + (j + 1)\Delta_c).
\]

Hence \( q \) oscillates back and forth in the configuration space along the arc \( q_c([\overline{\alpha}_c, \overline{\omega}_c]) \), it verifies \( W(q_c(t)) > c \) for any \( t \in (\overline{\alpha}_c, \overline{\omega}_c) \), and it has period \( 2(\overline{\omega}_c - \overline{\alpha}_c) \). Such solution constructed as in (3.9) is said to be a brake type orbit.

### 3.1.2 Statement of the problem and the main Theorem 3.1

The main goal of this chapter is to establish a convergence result for a sequence of brake type orbits \( \{q_c\} \), as \( c \to 0^+ \), each one solving (3.1) and bridging \( \{ W \leq c \}^- \) to \( \{ W \leq c \}^+ \), to a classical solution \( U_0 \) the \textit{heteroclinic connection equation} between the wells of \( W \),

\[
U_0'' = \nabla W(U_0) \quad \text{in } \mathbb{R}, \quad U_0(-\infty) = p_-, \quad U_0(+\infty) = p_+.
\]

This limiting solution \( U_0 \) turns out to be a minimizer of the \textit{heteroclinic variational problem}

\[
m_0 := \inf\{ H^0(V) : V \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \}, \tag{HCP}
\]

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where the admissible class is an affine Sobolev space $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) := U_{\text{aff}} + H^1(\mathbb{R}; \mathbb{R}^N)$, and $U_{\text{aff}}$ is an affine function connecting the two wells, extended trivially to $\mathbb{R}$:

$$
U_{\text{aff}}(t) = \left( \frac{(1-t)}{2} p_- + \frac{(1+t)}{2} p_+ \right) \chi_{(-1,1)}(t) + p_- \chi_{(-\infty,-1]}(t) + p_+ \chi_{[1,\infty)}(t).
$$

The existence of such $H^0$-minimizing heteroclinic is guaranteed under reasonable assumptions on the potential, to be made precise later, by one of the main result in Chapter 2 since $W$ is a double-well potential (see Corollary 1.2).

One can formally regard the heteroclinic variational problem (HCP) as a limit when $c \to 0^+$ of the variational problems (BTPc) which give rise to brake type orbits. The primary goal of this chapter consists of a rigorous study of a relationship between such variational problems.

In order to carry out the analysis one needs, in some sense, to extend the trajectories of brake type orbits up to the wells of $W$, i.e. the components of the zero-level set of $W$. In §3.3 we devise a method that for small values of $c$ allows us to relate admissible curves $q_c \in \mathcal{M}_c$ for the variational problem (BTPc) “at level $c$” with admissible curves to the heteroclinic problem (HCP) “at level zero”. Given $c > 0$ and a minimizer $q_c$ of (BTPc), we create a new curve $q_{c,2c} \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ by modifying $q_c$ in the following manner: The set of times where $q_c$ lies on the superlevel set $\{W > 2c\}$ is shown to be a bounded open interval $(\tau_{2c}^-, \tau_{2c}^+)$. In this way, the new curve equals $q_c$ on $(\tau_{2c}^-, \tau_{2c}^+)$, however on $(-\infty, \tau_{2c}^-)$ and $(\tau_{2c}^+, +\infty)$ we define $q_{c,2c}$ as the solution to the corresponding gradient flow initial value problem

$$
\pm \Phi'(t) = -\nabla W(\Phi(t)) \quad \text{for} \quad \pm t \geq 0, \quad \Phi(0) = q_c(\tau_{2c}^\pm).
$$

In this way, we have effectively constructed a curve that connects the wells $p_-$ and $p_+$, with finite energy $H^0$. For details we refer the reader to the definitions in §3.3.

**Remark 3.1.** The “natural” extension of the minimizer $q_c : [\alpha_c, \omega_c] \to \mathbb{R}^N$ to the wells using gradient flow, starting from contact times at $\{W = c\}^\pm$, has not shown to be as effective as the given one, mainly due to technical difficulties in our analysis.
In order to make the last procedure work, including the very existence of the family of minimizers \( \{q_c\} \), we need additional hypotheses on the properties that the potential enjoys. We emphasize, however, most of these are standard in the literature.

\( (W1) \) \( W \in C^3(\mathbb{R}^N) \) and \( p_{\pm} \) are the only global minima at zero, \( W(p_{\pm}) = 0 \). Furthermore,
\[
\liminf_{|p| \to \infty} W(p) > 0.
\]
This assumption, among other things, ensures that the value \( m_c := \inf \{H^c(q) : q \in \mathcal{M}_c\} \), for sufficiently small \( c > 0 \), does not increase when minimizing over a reduced admissible set with \( L^\infty \)-constraint. In other words, there is \( R_0 > 0 \) so that
\[
m_c = \inf \{H^c(q) : q \in \mathcal{M}_c, \|q\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} \leq R_0\}.
\]

\( (W2) \) There exist \( 0 < \lambda \leq \Lambda < \infty \) so that
\[
\lambda I_{N \times N} \leq D^2W(p_{\pm}) \leq \Lambda I_{N \times N}.
\]
Let us denote the linear segment in \( \mathbb{R}^N \) between the points \( p_1 \) and \( p_2 \) by
\[
[p_1, p_2] := \{p_\lambda \in \mathbb{R}^N : p_\lambda = (1 - \lambda)p_1 + \lambda p_2 \text{ for some } \lambda \in [0, 1]\}.
\]
Suppose in addition, there exists \( c_0 \in (0, \max[p_-, p_+] W) \) in such a way that the following properties hold

\( (W3) \) \( \forall c \in (0, c_0) : \mathcal{W}_c \) is partitioned into disjoint sets \( \mathcal{W}_c^- = \{W \leq c\}^- \), \( \mathcal{W}_c^+ = \{W \leq c\}^+ \), each enclosing \( p_- \) and \( p_+ \), respectively.

\( (W4) \) Every \( c \in (0, c_0) \) is a regular value for \( W \), i.e., \( \bigcup_{c \in (0, c_0)} \{W = c\} \subset \{\nabla W \neq 0\} \).

Now we are ready to introduce the main result of the second portion of Part I:

**Theorem 3.1.** Assume \( W \) satisfies \((W1)\) through \((W4)\). Then for any sequence \( c_n \to 0^+ \) the family of variational problems \( \{(\text{BTP}_{c_n})\} \) approaches the variational heteroclinic connection problem (HCP), in the following senses:

For any sequence \( \{q_{c_n}\} \) of minimizers to \( (\text{BTP}_{c_n}) \), there holds
(i) \( \lim_{n \to \infty} m_{cn} = m_0 \), where \( m_{cn} := H^{cn}(a_{cn}) \) and \( m_0 := \inf \{ H^0(V) : V \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \} \).

(ii) There exist times \( \{ t_k \} \subset \mathbb{R} \), and a subsequence \( \{ a_{cn_k, 2cn_k} \} \) of the gradient flow extensions from the level \( 2cn_k \) of the minimizers \( \{ a_{cn} \} \) (see Definition 3.3) such that for \( a_k := a_{cn_k, 2cn_k} (\cdot - t_k) \) one has

\[
a_k - U_0 \rightharpoonup 0 \quad \text{weakly in} \quad H^1(\mathbb{R}; \mathbb{R}^N) \quad \text{as} \quad k \to \infty,
\]

where \( U_0 \) is a minimizer of \( (\text{HCP}) \), in particular, a heteroclinic connection between the wells of \( W \).

The reader is encouraged to revisit Remark 1.2, Remark 1.3, and Remark 1.4, made earlier in Chapter 1 of this thesis, for a detailed description about the scope of this theorem.

We start our analysis in §3.2 by remarking that a renormalization in time (translation) of the sequence \( \{ a_c \} \) may be necessary to deduce some properties of the potential values function \( t \mapsto W(a_c(t)) \), the new sequence will be denoted by \( \{ a_c \} \). Procedures such as extension by gradient flow and truncation of curves at level sets of \( W \) are discussed in §3.3, with the purpose of relating the admissible classes \( H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \) and \( \mathcal{M}_c \). In §3.4 we get bounds on the \( H^0 \) energies of \( \{ a_c, 2c \} \), and show they are almost optimal for \( (\text{BTP}_c) \). Later in §3.5 we study the limit of the optimal values \( \{ m_c \} \) as \( c \to 0^+ \). As it turns out, this limit equals \( m_0 \), the optimal value for the limiting heteroclinic problem. This property is enough to prove that the sequence \( \{ a_{c, 2c} \} \) is minimizing for the limiting variational problem \( (\text{HCP}) \), as \( c \to 0^+ \). The full proof of Theorem 3.1 is established in §3.6 where we show the boundedness of \( a_{c, 2c} \) in \( H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \) as \( c \to 0^+ \), through the study of the convergence of \( a_{c, 2c} \) to the wells for times outside of a large (but fixed) compact interval. Then a soft analysis argument will render the desired convergence of \( \{ a_{c, 2c} \} \) to a minimizing heteroclinic connection.
3.2 Qualitative properties of minimizing brake type orbits

Let us recall some notation and properties from the discussion in §3.1.1 of the minimizers that give rise to the brake type orbit solutions in [6]. For every \( c \in (0, c_0) \) as in \((W3)-(W4)\), \( -\infty < \overline{c} < \overline{c} < +\infty \) are the times associated to a minimizer \( q_c \) of \((BTP_c)\) for which

\[
\begin{align*}
  m_c := H^c(q_c) &= H^c(q_c|_{[\overline{c}, \overline{c}]}) , \\
  q_c(t) &\in \{ W > c \} \quad \text{for} \quad t \in (\overline{c}, \overline{c}) , \\
  q_c(t) &\equiv q_c(\overline{c}) \quad \text{for all} \quad t < \overline{c} \quad \text{and} \quad q_c(\overline{c}) \in \{ W = c \}^- , \\
  q_c(t) &\equiv q_c(\overline{c}) \quad \text{for all} \quad t > \overline{c} \quad \text{and} \quad q_c(\overline{c}) \in \{ W = c \}^+ .
\end{align*}
\]

so the minimizer effectively bridges the two components \( \{ W \leq c \}^- \) and \( \{ W \leq c \}^+ \) of the \( c \)-sublevel set of the potential in a finite time.

The desired convergence analysis of the sequence of brake type orbits essentially relies on a quantitative description of the behavior the minimizers \( t \mapsto q_c(t) \) of \((BTP_c)\) on small level sets of the potential \( \{ W = \beta \} \) for \( \beta \) close to \( c \). We start by pointing out some crucial properties the sublevel sets \( \{ W \leq \beta \} \) sets enjoy, for small values of \( \beta \).

3.2.1 Geometric description of the level sets of \( W \)

Let us highlight a couple of geometric and topological properties of the level sets of \( W \) that are close to the wells, which are a consequence of the above assumptions. First, \((W1)\) and \((W4)\) imply that level sets of the potential that are close to the wells cannot be have interior, namely, \( \partial \{ W = c \}^\pm = \{ W = c \}^\pm \) for any \( c \in (0, c_0] \). In addition, by expanding \( W \) in a Taylor development around the wells and using the quadratic non-degeneracy of \( W \) at the wells \((W2)\), plus the fact that \( \{ W \leq c \}^\pm \) shrinks down to \( p^\pm \) as \( c \to 0^+ \), we obtain there exists \( \sigma_0 > 0 \) so that

\[(W2^*) \quad \quad W(u) \geq \sigma_0 |u - p^\pm|^2 \quad \text{for any} \quad u \in \{ W \leq c_0 \}^\pm ,\]

where the value of \( c_0 > 0 \) as in \((W3)-(W4)\) may have to be reduced, if necessary. This non-degeneracy of the potential can be used, moreover, not only to argue that the level sets of \( W \) near
the wells are trapped between two Euclidean balls centered at each well \( p_{\pm} \), respectively, but also to show that the value of \( |\nabla W|^2 \) is comparable to that of \( W \) on these level sets. The next lemma gives a more quantitative description of the above properties.

**Lemma 3.1 (Level sets near the wells).** There exist constants \( \varkappa > \kappa > 0 \) and \( c_1 > 0 \), depending on the dimension \( N \), on \( c_0, \sigma_0 \) as in \((W2^*)\), on \( \lambda, \Lambda \) as in \((W2)\), on \( |p_+ - p_-| \), and on bounds of \( D^3W \) near the wells, in such a way that

\[
\frac{1}{\Lambda} W(u) \leq |u - p_{\pm}|^2 \leq \frac{3}{\lambda} W(u), \quad (3.12)
\]

\[
\kappa W(u) \leq |\nabla W(u)|^2 \leq \varkappa W(u), \quad (3.13)
\]

hold for any \( u \in \{W \leq c_1\}^\pm \).

Even though the proof of this result is elementary, we include it for sake of completeness.

**Proof of Lemma 3.1.** Let us start by defining

\[
\nu := \sup \{|D^3W(p)| : p \in \overline{B_{r_0}(p_+)} \cup \overline{B_{r_0}(p_-)}\} \quad \text{for} \quad r_0 := \frac{1}{2}|p_+ - p_-|. \quad (3.14)
\]

Throughout, we fix \( u \in \mathbb{R}^N \) in one of the components of the \( c_1 \)-sublevel set of the potential \( \{W \leq c_1\}^+ \), for a value of \( c_1 \) chosen appropriately small later in the proof. The other case where \( u \in \{W \leq c_1\}^- \) can be argued similarly. To establish (3.12), let us consider the Taylor development of \( W \) around the well \( p_+ \),

\[
W(u) = 0 + 0 + \frac{1}{2}D^2W(p_+)(u - p_+) \cdot (u - p_+) + R(u - p_+), \quad (3.15)
\]

where the remainder \( R(x) \) satisfies the bound \(|R(x)| \leq (\nu N^3/6)|x|^3 \) for any \( x \in B_{r_0}(0) \). Letting \( c_1 \leq c_0 \) we get from the property \((W2^*)\) of \( W \) that any \( u \in \{W \leq c_1\}^+ \) satisfies

\[
|u - p_+|^2 \leq \frac{W(u)}{\sigma_0} \leq \frac{c_1}{\sigma_0}, \quad (3.16)
\]

If we further require \( c_1 \leq \sigma_0 \min\{r_0^2, (\lambda/\nu N^3)^2\} \), then in view of (3.16) the inequalities \( |u - p_+| \leq r_0 \) and \( |u - p_+| \leq \lambda/N^3 \nu \) hold, which imply \(|R(u - p_+)\| \leq (\lambda/6)|u - p_+|^2 \) due to the definition (3.14).
The latter inequality combined with the Taylor expansion (3.15) and the property (W2) of the potential, yields altogether

\[ W(u) \leq \frac{\Lambda}{2} |u - p_+|^2 + \frac{\lambda}{6} |u - p_+|^2 \leq \Lambda |u - p_+|^2, \quad \text{and} \]

\[ W(u) \geq \frac{\lambda}{2} |u - p_+|^2 - \frac{\lambda}{6} |u - p_+|^2 \geq \frac{\lambda}{3} |u - p_+|^2. \]

(3.17)

On the other hand, taking the Taylor development (3.15) of \( \nabla W \) around \( p_+ \) we obtain

\[ \nabla W(u) = D^2 W(p_+)(u - p_+) + R(u - p_+), \]

where the error is \( R^i(x) = \int_0^1 \left( \sum_{j,k=1}^N D^{(i,j,k)} W(p_+ + tx)x^j x^k \right) \frac{(t - 1)^2}{2} dt, \) for all \( 1 \leq i \leq N \) and \( x \in \mathbb{R}^N. \) Provided \( x \in B_{r_0}(0), \) the estimate \( |R(x)| \leq \nu N|x|^2 \) holds with \( \nu = \nu(D^3 W) \) as in (3.14).

In particular, this estimate holds for \( u - p_+ \in B_{r_0}(0). \)

Let us now observe,

\[ |\nabla W(u)|^2 = |D^2 W(p_+)(u - p_+)|^2 + 2D^2 W(p_+)(u - p_+) \cdot R(u - p_+) + |R(u - p_+)|^2, \]

for which the inequalities in (3.17) and the estimate of \( |R(u - p_+)| \) apply, to yield

\[ |\nabla W(u)|^2 \geq \left( \lambda^2 - 2\Lambda N \nu |u - p_+| \right) |u - p_+|^2 \]

\[ \geq \left( \lambda^2 - 2\Lambda N \nu \sqrt{\frac{c_1}{\sigma_0}} \right) \frac{1}{\Lambda} W(u) \geq \frac{\lambda^2}{2\Lambda} W(u). \]

\[ =: \kappa(\lambda, \Lambda) W(u), \quad \text{provided} \quad c_1 \leq \left( \frac{\lambda^2 \sqrt{\sigma_0}}{4\Lambda N \nu} \right)^2 \text{ is small enough}. \]

Analogously, one derives using the same estimates as above,

\[ |\nabla W(u)|^2 \leq (\Lambda^2 + 2\Lambda N \nu |u - p_+| + N^2 \nu^2 |u - p_+|^2) |u - p_+|^2 \]

\[ \leq \left( \Lambda^2 + 2\Lambda N \nu \sqrt{\frac{c_0}{\sigma_0}} + N^2 \nu^2 \frac{c_0}{\sigma_0} \right) \frac{3}{\Lambda} W(u) \]

\[ =: \kappa(N, \lambda, \Lambda, c_0, \nu) W(u), \]

thus concluding the proof of the lemma.

The analysis of the qualitative behavior of the brake type orbits \( \{q_s\} \) near the wells \( p_{\pm}, \) relies in part on the properties enjoyed by \( \Phi_1, \Phi_2, \) defined as \( \mathbb{R}^N \)-valued solutions to the following gradient
flow system of $W$ with arbitrary initial data $b^- \in \{ W \leq c_0 \}^-$ and $b^+ \in \{ W \leq c_0 \}^+$, respectively,

$$
\frac{d}{dt} \Phi_1(t, b^-) = \nabla W(\Phi_1(t, b^-)) \quad \text{for } t < 0, \quad \Phi_1(0, b^-) = b^-.
$$

(GF₁)

$$
\frac{d}{dt} \Phi_2(t, b^+) = -\nabla W(\Phi_2(t, b^+)) \quad \text{for } t > 0, \quad \Phi_2(0, b^+) = b^+.
$$

(GF₂)

**Remark 3.2.** The maximal interval of existence of $t \mapsto \Phi_j(t, u_0)$ contains $(-\infty, 0]$ for $j = 1$ and $[0, +\infty)$ for $j = 2$, respectively, provided the initial data $u_0$ lies near $p_-$ or $p_+$.

The hypotheses on $W$ ensure an exponential rate of convergence of the solution to the gradient flow system to the corresponding well. That is,

**Lemma 3.2 (Gradient flow decay).** For the constant $c_1$ of Lemma 3.1, consider any initial data $b^+ \in \{ W \leq c_1 \}^+$. Then, the solutions $\Phi_j$ to (GF₉), for $j = 1, 2$, satisfy

$$
\sup_{t \in (-\infty, 0]} e^{-\kappa t} W(\Phi_1(t, b^-)) \leq W(b^-), \quad \sup_{t \in [0, +\infty)} e^{\kappa t} W(\Phi_2(t, b^+)) \leq W(b^+).
$$

where $\kappa > 0$ is the constant in Lemma 3.1.

**Proof of Lemma 3.2.** We argue the inequality for $W(\Phi_2(t, b^+))$ for $t \in [0, \infty)$, and to simplify the notation we drop the index $j = 2$. The other inequality is derived similarly. Consider $\phi(t) := W(\Phi(t, b^+))$, then $\phi'(t) = -|\nabla W(\Phi(t, b^+))|^2 \leq 0$ so $\phi(t) \leq \phi(0) = W(b^+) = \beta$ for all $t \geq 0$, that is, $\Phi(t, b^+) \in \{ W \leq \beta \}^+$ for $t \geq 0$. Since $\beta \leq c_1$, the comparison Lemma 3.1-(3.13) yields

$$
\phi'(t) = -|\nabla W(\Phi(t, b^+))|^2 \leq -\kappa W(\Phi(t, b^+)) = -\kappa \phi(t),
$$

for all $t \geq 0$. The differential inequality implies $\phi(t) \leq \phi(0)e^{-\kappa t}$ for any $t \in [0, +\infty)$. \hfill \Box

This rapid rate of convergence yields the following integral bounds

**Corollary 3.1.** Let $b^+ \in \{ W \leq c_1 \}^+$ and $\Phi_j$, for $j = 1, 2$, be as in Lemma 3.2. Then

$$
\int_{-\infty}^{0} \frac{1}{2}|\Phi'_1(t, b^-)|^2 + W(\Phi_1(t, b^-)) \, dt \leq \frac{\kappa + 2}{2\kappa} W(b^-),
$$

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and
\[
\int_0^{+\infty} \frac{1}{2} |\Phi'_2(t, b^+)|^2 + W(\Phi_2(t, b^+)) \, dt \leq \frac{\kappa + 2}{2\kappa} W(b^+).
\]

where \(\kappa, \zeta\) are the constants in Lemma 3.1.

**Proof of Corollary 3.1.** We only argue the bound for \(\Phi_2\), the other case is analogous. Since \(\Phi_2\) solves (GF_2) and also that \(\Phi_2(t, b^+) \in \{W \leq c_1\}^+\) for all \(t > 0\) imply that the comparison Lemma 3.1 yields the pointwise bound
\[
|\Phi'_2(t, b^+)|^2 = |\nabla W(\Phi_2(t, b^+))|^2 \leq \zeta W(\Phi_2(t, b^+)) \quad \text{for all } t > 0.
\]

From this and the exponential decay in Lemma 3.2, we conclude
\[
\int_0^{+\infty} \frac{1}{2} |\Phi'_2(t, b^+)|^2 + W(\Phi_2(t, b^+)) \, dt \leq \frac{\zeta + 2}{2} \int_0^{+\infty} W(\Phi_2(t, b^+)) \, dt
\]
\[
\leq \frac{\zeta + 2}{2} \int_0^{+\infty} W(b^+) e^{-\kappa t} \, dt.
\]

\[\square\]

### 3.2.2 Renormalization of the brake type orbits and time control

Let us now make an observation that will play an important role in our construction.

**Lemma 3.3.** There exists a sequence of times \(\{t_c\}\), in such a way that the time translations
\[q_c := q_c(\cdot - t_c)\]

of the brake type orbits minimizers satisfy the uniform bound at time zero:
\[
\inf_{c \in (0, c_0)} W(q_c(0)) \geq c_0 > 0,
\]

where \(c_0\) is the constant appearing in the properties (W3) through (W4) of \(W\).

**Proof of Lemma 3.3.** This can be done, for example, if we make a choice of distinguished times \(\{t_c\}\) as follows:
\[
\text{For every } 0 < c < c_0, \text{ select } t_c \in \mathbb{R}, \text{ so that } W(q_c(t_c)) > c_0.
\]
Having (3.19) at our disposal, the validity of Lemma 3.3 follows immediately. Now we just need to argue that the selection process (3.19) can always be carried out. Indeed, arguing by contradiction, suppose \( \exists c' \in (0, c_0) \) so that
\[
\sup\{W(q_{c'}(t)) : t \in [\overline{\alpha}_{c'}, \overline{\omega}_{c'}]\} \leq c_0.
\] (3.20)
Because \( q_{c'} \) is a minimizer of (BTP\(c'\)) then it follows from (3.11) that \( q_{c'}(\overline{\alpha}_{c'}) \in \{W = c'\}^- \subset \{W \leq c_0\}^- \). This fact combined with the contradiction hypothesis (3.20) and the assumption (W3) on the level sets of \( W \), implies that \( q_{c'}([\overline{\alpha}_{c'}, \overline{\omega}_{c'}]) \subset \{W \leq c_0\}^- \). However, this immediately contradicts the fact that \( q_{c'}(\omega_{c'}) \in \{W = c'\}^+ \subset \{W \leq c_0\}^+ \), by (3.11) once again.

In light of the previous lemma, let us define the sequence of time translations of brake type orbits minimizers,
\[
\alpha_c := \overline{\alpha}_c + t_c, \quad \omega_c := \overline{\omega}_c + t_c,
\]
\[
q_c : [\alpha_c, \omega_c] \to \mathbb{R}^N, \quad q_c(t) := q_c(t - t_c) \text{ for } t_c \text{ as in (3.19)}.
\] (3.21)

**Note.** In the remainder of this chapter we will work with the sequence of time translations \( \{q_c\} \) in (3.21), rather the original sequence of brake type orbits \( \{q_c\} \). Nonetheless, the reader will be reminded about the normalization (3.21) in the proof of the main result of this chapter, Theorem 3.1.

**Remark 3.3.** It is in the sense of (3.18) that the translation in time (3.21) has a vital “renormalizing” property on the intervals of definition
\[
0 \in \bigcap_{0 < c < c_0} (\alpha_c, \omega_c),
\] (3.22)
that ultimately allows us to show a convergence of the family of brake type orbits. This renormalization has a “compactification” effect on the sequence; see the discussion in the paragraph following equation (1.6). Having this issue in mind, one can understand the effect of condition (3.22), since it allows us to obtain upper bounds on \( \{\alpha_c\} \) and \( \{\omega_c\} \), provided we know a bound on the measure of the intervals \( (\alpha_c, \omega_c) \), which is much easier to derive:
\[
\limsup_{c \to 0} |(\alpha_c, \omega_c)| < M \quad \text{implies} \quad \limsup_{c \to 0} \max\{|\alpha_c|, |\omega_c|\} < M.
\]
We point out since the system (3.1) and the functional \( H^c \) are invariant under translations in time, we immediately get from (3.23) that the sequence of translations (3.21) satisfies

\[
m_c = H^c(q_c) = H^c(q_c|_{[\alpha_c, \omega_c]}),
q_c(t) \in \{ W > c \} \text{ for } t \in (\alpha_c, \omega_c),
q_c(t) \equiv q_c(\alpha_c) \text{ for all } t < \alpha_c \text{ and } q_c(\alpha_c) \in \{ W = c \}^-, 
q_c(t) \equiv q_c(\omega_c) \text{ for all } t > \omega_c \text{ and } q_c(\omega_c) \in \{ W = c \}^+.
\]

(3.23)

In the sections to come, we consider a family of times which we denote by \( \tau_{c,\beta}^\pm \), that will be helpful in performing a sensible asymptotic analysis on the brake type orbit family \( \{ q_c \} \) for \( c \sim 0 \).

**Definition 3.1 (Control times on level sets).** For any two numbers \( c \) and \( \beta \) satisfying

\[
0 < c < \beta \leq \min\{c_0, c_1\},
\]

(3.24)

with \( c_1 \) as in Lemma 3.1, we define two times \( \tau_{c,\beta}^- \) and \( \tau_{c,\beta}^+ \) in such a way that \( (\tau_{c,\beta}^-, \tau_{c,\beta}^+) \) is the largest open interval containing 0 on which \( q_c(t) \in \{ W > \beta \} \). That is to say,

\[
\tau_{c,\beta}^- := \inf\{ t \in [\alpha_c, 0] : \min_{s \in [t, 0]} W(q_c(s)) > \beta \},
\tau_{c,\beta}^+ := \sup\{ t \in [0, \omega_c] : \min_{s \in [0, t]} W(q_c(s)) > \beta \}.
\]

(3.25)

We say that \( \tau_{c,\beta}^- \) and \( \tau_{c,\beta}^+ \) are control times for \( q_c \) on the superlevel set \( \{ W > \beta \} \).

The continuity of \( W \) together with condition (3.18), and the definitions of \( \alpha_c, \omega_c \) ensure that the times \( \tau_{c,\beta}^\pm \) are well defined for any \( c, \beta \) satisfying (3.24), and moreover

\[
q_c(\tau_{c,\beta}^-) \in \{ W = \beta \}^-, \quad q_c(\tau_{c,\beta}^+) \in \{ W = \beta \}^+,
\tau_{c,\beta}^- := \lim_{\beta \to c^+} \tau_{c,\beta}^- = \alpha_c, \quad \tau_{c,\beta}^+ := \lim_{\beta \to c^+} \tau_{c,\beta}^+ = \omega_c,
\alpha_c < \tau_{c,\beta}^- < 0 < \tau_{c,\beta}^+ < \omega_c.
\]

**Remark 3.4.** An immediate consequence of Definition 3.1 is the following monotonicity property of \( \beta \mapsto \tau_{c,\beta}^\pm \), keeping \( c \) fixed: For any \( 0 < c < \beta_1 < \beta_2 \leq \min\{c_0, c_1\} \), there holds

\[
\tau_{c,\beta_1}^- < \tau_{c,\beta_2}^- < 0 < \tau_{c,\beta_2}^+ < \tau_{c,\beta_1}^+.
\]

(3.26)
Before embarking on this construction in detail, however, we would like to remark on the qualitative behavior of the minimizer \( q_c \) of (BTPc) near its contact points in \( \{ W = c \}^\pm \). We will essentially show that during the times where the curve \( q_c(t) \) lies in \( \{ c < W < c^* \}^\pm \), for some fixed value of \( c^* \), one has that \( q_c \) transverses all the level sets of \( W \) in a monotonic fashion: increasingly for \( t \) close to \( \alpha_c \), and decreasingly for \( t \) close to \( \omega_c \).

**Proposition 3.1 (Behavior of brake orbits on level sets of \( W \)).** There exists a positive constant \( c^*_* \) depending only on \( W \), so that, for every choice of \( c \in (0, c^*_*) \),

(i) The mapping \( t \mapsto W(q_c(t)) \) is a strictly increasing convex function for \( t \in (\alpha_c, \tau_c^-c^* \}].

(ii) The mapping \( t \mapsto W(q_c(t)) \) is a strictly decreasing convex function for \( t \in [\tau_c^+, \omega_c) \).

In particular, \( q_c \) crosses \( \{ W = \beta \}^- \) and \( \{ W = \beta \}^+ \) precisely once for every \( \beta \in (c, c^*) \) and it never does so tangentially, for such range of \( c \).

**Proof of Proposition 3.1.** We will argue that there is a constant \( 0 < c^*_+ \leq \min\{c_0, c_1\} \), for the constants appearing in (3.24), so that

\[
\forall c \in (0, c^*_+), \varphi_c(t) := W(q_c(t)) \text{ is a strictly decreasing convex function in } [\tau_c^+, \omega_c). \tag{3.27}
\]

The existence of the analogous \( c^*_- \) for (i) is argued similarly. Then the constant \( c^*_* := \min\{c^*_+, c^*_-\} \in (0, \min\{c_0, c_1\}) \) will satisfy (i)-(ii) simultaneously.

Let us assume by contradiction there is no \( c^*_+ > 0 \) satisfying (3.27), so that we can find sequences \( \beta_n \to 0^+, c_n \to 0^+ \) with \( 0 < c_n < \beta_n \) for which \( \varphi_{c_n} \) fails to be decreasing on the whole interval \( [\tau_{c_n, \beta_n}^+, \omega_{c_n}) \); namely, there is \( \{t_n\} \subset \mathbb{R} \) with \( \tau_{c_n, \beta_n}^+ \leq t_n < \omega_{c_n} \) and

\[
\varphi'_{c_n}(t_n) = \nabla W(q_{c_n}(t_n)) \cdot q'_{c_n}(t_n) \geq 0.
\]

We will establish a contradiction by following the steps outlined below. For \( n \) large enough, we are going to show that
I. \( \varphi_{cn} \) is strictly convex on \( \{ t : q_{cn}(t) \in \{ W \leq \hat{c} \} \} \), for \( \hat{c} > 0 \) small and fixed.

II. The choice of \( t_n \) can be made so \( \varphi'_{cn}(t_n) = 0 \).

III. \( q_{cn} \) must exit the set \( \{ W \leq \hat{c} \} \), before reaching the terminal time \( t = \omega_{cn} \).

IV. The behavior of \( q_{cn} \) in III is energetically expensive, in that, the minimality of \( H_{cn}(q_{cn}) \) over \( M_{cn} \) fails to be true. If \( I_n \subset [\tau_{cn, \beta_n, \omega_{cn}}] \) is a connected component of the set of times \( \{ t : \hat{c}/2 < \varphi_{cn}(t) < \hat{c} \} \), then we argue in two steps:

IV.1. Showing that \( \liminf_{n \to \infty} H_{I_n}^{cn}(q_{cn}) > 0 \).

IV.2. Building a competitor in \( M_{cn} \) with less energy than \( q_{cn} \), by replacing the restriction \( q_{cn}|_{I_n} \) with a linear segment \( \gamma_{aff} \) near the well \( p_+ \), in such a way that \( H_{cn}(\gamma_{aff}) = O(\beta_n) \to 0 \) as \( n \to \infty \).

To prove I, we note the quadratic non-degeneracy (W2) of \( W \) implies \( \exists \hat{c} \in (0, c_0) \) s.t.

\[
D^2 W(p) > \frac{\lambda}{2} I_{N \times N} \quad \text{for all } p \in \{ W \leq \hat{c} \}. \quad (3.28)
\]

The strict convexity of the function \( \varphi_{cn} \) on \( \{ t : q_{cn}(t) \in \{ W \leq \hat{c} \} \} \) is a consequence of the ODE solved by \( q_{cn} \), and the fact that (3.28) remains valid in this set of times. Indeed,

\[
\varphi''_{cn}(t) = D^2 W(q_{cn})q'_{cn}(t)q''_{cn}(t) + \nabla W(q_{cn}) \cdot q''_{cn}(t) \\
\geq \frac{\lambda}{2} |q''_{cn}(t)|^2 + |\nabla W(q_{cn})|^2 \\
\geq \min\{ |\nabla W(p)|^2 : p \in \{ c_n \leq W \leq \hat{c} \} \} > 0,
\]

where the positivity of the last quantity follows from property (W4). The proof of I is now complete.

For II, we claim that in the event \( t_n \) satisfies \( \varphi'_{cn}(t_n) > 0 \) then we can choose another time \( t'_n \in (\tau_{cn, \beta_n, \omega_{cn}}) \) with \( \varphi'_{cn}(t'_n) = 0 \). Indeed, from \( \varphi_{cn}(\omega_{cn}) - \varphi_{cn}(\tau_{cn, \beta_n}) = c_n - \beta_n < 0 \) we deduce using the mean value theorem that \( \exists \xi_n \in (t_n, \omega_{cn}) \) with \( \varphi'_{cn}(\xi_n) < 0 \). Then the continuity of \( \varphi'_{cn} \) allows us to use the intermediate value theorem to find a time \( t'_n \) between \( \xi_n \) and \( t_n \), at which this
derivative vanishes. After renaming \( t_n := t'_n \) if necessary, we have found \( t_n \in (\tau^{+}_{c_n, \beta_n}, \omega_{c_n}) \) with \( \varphi'_{c_n}(t_n) = 0 \) for any \( n \geq 1 \), thus proving the claim.

To argue III, we note on the one hand that \( \varphi'_c(t_n) = 0 \) and also from the Hamiltonian identity

\[
|q'_c(t)|^2 = 2(\varphi_c(t) - c_n) \text{ that } q'_c(\omega_{c_n}) = 0, \text{ so } \varphi''_c(\omega_{c_n}) = \nabla W(q_{c_n}(\omega_{c_n})) \cdot 0 = 0.
\]

The mean value theorem then shows \( \exists \), for any \( n \geq 1 \), thus proving the claim.

For point IV let us observe that because \( q_{c_n} \) develops a finger, as established in III, there always exists an open interval \( I_n \subset (t_n, \omega_{c_n}) \) such that \( \hat{c}/2 < \varphi_{c_n} < \hat{c} \) on \( I_n \), with \( \min_{\partial I_n} \varphi_{c_n} = \hat{c}/2 \) and \( \max_{\partial I_n} \varphi_{c_n} = \hat{c} \), see Figure 3.1 below.

![Figure 3.1: Finger formation in \( q_{c_n} \) in the contradiction argument of Proposition 3.1.](image)

The Cauchy-Schwartz inequality yields,

\[
H_{I_n}^{c_n}(q_{c_n}) \geq \int_{I_n} \sqrt{W(q_{c_n}) - c_n} |q'_c| \, dt \geq \sqrt{\frac{\hat{c}}{4}} \text{dist}(\{W = \hat{c}/2\}^+, \{W = \hat{c}\}^+) =: \vartheta,
\]

provided \( c_n < \hat{c}/4 \) for \( n \) large enough. Hence IV.1 has been established:

\[
\liminf_{n \to \infty} H_{I_n}^{c_n}(q_{c_n}) \geq \vartheta > 0.
\]
The proof of Proposition 3.1 will be complete once we prove IV.2, as this would show that for $n$ sufficiently large $q_{cn}$ is not a minimizer of $H^{cn}$ over $\mathcal{M}_{cn}$.

Let us consider $\Phi_2 : [0, \infty) \to \mathbb{R}^N$ as introduced in §3.2.1, the solution to the gradient flow of $W$ with initial data $b^+ = P_n$. That is,

$$\frac{d}{dt} \Phi_2(t, P_n) = -\nabla W(\Phi_2(t, P_n)) \quad \text{for} \quad t > 0, \quad \Phi_2(0, P_n) = P_n,$$

where we choose $P_n := q_{cn}(\tau_{cn, \beta_n}^+, \tau_{cn, \beta_n}^+ + \hat{t}_n)$. Let us recall from the proof of the decay Lemma 3.2 that $t \mapsto W(\Phi_2(t, P_n))$ is a strictly decreasing function on $(0, \infty)$, by virtue of property (W4) of small level sets of $W$. Thus, there is a unique time $\hat{t}_n > 0$ with the property that $\Phi_2(\hat{t}_n, P_n) \in \{W = c_n\}^+$. Let us recall from the proof of the decay Lemma 3.2 that $t \mapsto W(\Phi_2(t, P_n))$ is a strictly decreasing function on $(0, \infty)$, by virtue of property (W4) of small level sets of $W$. Thus, there is a unique time $\hat{t}_n > 0$ with the property that $\Phi_2(\hat{t}_n, P_n) \in \{W = c_n\}^+$; call $Q_n$ such a point (see Figure 3.1 above).

We now build the competitor $\tilde{q}_n$ in $\mathcal{M}_{cn}$, by letting

$$\tilde{q}_n(t) := \begin{cases} 
q_{cn}(\alpha_{cn}), & \text{for } t \in (-\infty, \alpha_{cn}), \\
q_{cn}(t), & \text{for } t \in [\alpha_{cn}, \tau_{cn, \beta_n}^+), \\
\Phi_2(t - \tau_{cn, \beta_n}^+, \beta_n), & \text{for } t \in [\tau_{cn, \beta_n}^+, \tau_{cn, \beta_n}^+ + \hat{t}_n), \\
Q_n, & \text{for } t \in [\tau_{cn, \beta_n}^+ + \hat{t}_n, +\infty).
\end{cases}$$

As $\tilde{q}_n$ is a truncation, we readily see that

$$H^{cn}(\tilde{q}_n) = H^{cn}_{(\alpha_{cn}, \tau_{cn, \beta_n}^+, \hat{t}_n)}(\tilde{q}_n) = H^{cn}_{(\alpha_{cn}, \tau_{cn, \beta_n}^+, \hat{t}_n)}(q_{cn}) + H^{cn}_{(0, \hat{t}_n)}(\Phi_2(\cdot, P_n)), \quad \text{(3.31)}$$

after a performing a linear change of variables in the term $H^{cn}_{(\tau_{cn, \beta_n}^+, \tau_{cn, \beta_n}^+, \hat{t}_n)}(\tilde{q}_n)$. Here,

$$H^{cn}_{(0, \hat{t}_n)}(\Phi_2(\cdot, P_n)) = \int_0^{\hat{t}_n} \frac{1}{2} |\Phi_2'(t, P_n)|^2 + (W(\Phi_2(t, P_n)) - c) \, dt \leq \frac{(\kappa + 2)}{2\kappa} W(P_n),$$

as Corollary 3.1 shows. Taking $n$ large enough to ensure $\beta_n < \min\{\hat{c}/4, \partial \kappa/(\kappa + 2)\}$ we get

$$H^{cn}_{(0, \hat{t}_n)}(\Phi_2(\cdot, P_n)) \leq H^{cn}_{(\tau_{cn, \beta_n}^+, \omega_{cn})}(q_{cn}), \quad \text{(3.32)}$$

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where in the last step we used \( I_n \subset \subset (\tau^+_{\alpha_n,\beta_n},\omega_n) \). Thus if \( n \) is taken sufficiently large, (3.31) together with (3.32) implies the desired conclusion of IV.2,

\[
H^{c_n}(\tilde{q}_n) < H^{c_n}_{(\alpha_n,\omega_n)}(q_{c_n}) = H^{c_n}(q_{c_n}),
\]

and a contradiction is finally reached in arguing the monotonicity in (ii) of the mapping \( t \mapsto W(q_c(t)) \) over \([\tau^+_{c,\hat{c}},\omega_c])\), uniformly in \(0 < c < c_*\).

To finalize the argument in (ii), we now explain the convexity of these mappings. From the point above, \( t \mapsto W(q_c(t)) \) is a strictly decreasing function for \( t \in [\tau^+_{c,\hat{c}},\omega_c) \) for all choices of \(0 < c < c_*\). By shrinking the value of \( c_* \) if necessary (less than \( \hat{c} \)) this monotonicity reveals \( q_c(t) \in \{W \leq \hat{c}\} \) for all \( t \in [\tau^+_{c,\hat{c}},\omega_c) \), whence the validity of (3.29) remains true in this time interval.

As a corollary we get that the first hitting times, \( \tau^\pm_{c,\beta} \), of the orbit of \( q_c \) to the level sets \( \{W = \beta\}^\pm \) as defined in (3.25), are actually the only hitting times for small values of \( \beta \in (c, c_*) \).

The observation is the following

**Corollary 3.2 (Times of brake orbits on superlevel sets of \( W \)).** Let \( c_* \) be the constant in Proposition 3.1. For any choice of \( c, \beta \) such that

\[
0 < c < \beta \leq c_*, \quad (3.33)
\]

if \( q_c \) is a brake type orbit minimizer of the variational problem (BTPc), then every set

\[
\mathcal{X}_{c,\beta} := \{t \in [\alpha_c,\omega_c) : W(q_c(t)) > \beta\} \quad (3.34)
\]

is connected, and moreover

\[
\mathcal{X}_{c,\beta} = (\tau^-_{c,\beta}, \tau^+_{c,\beta}),
\]

where \( \tau^-_{c,\beta} \) and \( \tau^+_{c,\beta} \) are given by (3.25). Equivalently put,

\[
W(q_c(t)) \leq \beta \quad \text{if and only if} \quad t \in [\alpha_c, \tau^-_{c,\beta}] \cup [\tau^+_{c,\beta}, \omega_c]. \quad (3.35)
\]
Proof of Corollary 3.2. Consider any $0 < c < \beta \leq c_\ast$. In general, the definition of the times $\tau_{c,\beta}^\pm$ shows the trivial containment

$$(\tau_{c,\beta}^-, \tau_{c,\beta}^+) \subset \{ t \in [\alpha_c, \omega_c] : W(q_c(t)) > \beta \}.$$ 

On the other hand, Remark 3.4 yields the following chain of inequalities

$$\alpha_c < \tau_{c,\beta}^- < \tau_{c,\ast}^- < 0 < \tau_{c,\ast}^+ < \tau_{c,\beta}^+ < \omega_c,$$

which combined with Proposition 3.1 shows $W(q_c(t)) < \beta$ for every $t \in (\alpha_c, \tau_{c,\ast}^-) \cup (\tau_{c,\ast}^+, \omega_c)$. That is to say,

$$\{ t \in [\alpha_c, \omega_c] : W(q_c(t)) > \beta \} \subset (\tau_{c,\beta}^-, \tau_{c,\beta}^+),$$

thus completing the proof of Corollary 3.2. \qed

The next observation is immediate from Remark 3.4 and yet crucial for our investigation.

**Corollary 3.3 (Monotonicity of time spent on superlevel sets of $W$).** Consider $0 < c < \beta_1 < \beta_2 \leq c_\ast$ with $c_\ast$ as in Proposition 3.1. Then the sets of time where the curve $t \mapsto q_c(t)$ lies in the superlevel sets $\{ W > \beta_1 \}$ and $\{ W > \beta_2 \}$, respectively, (see (3.34)) satisfy the strict inclusion

$$\Sigma_{c,\beta_2} \subset \subset \Sigma_{c,\beta_1}.$$

### 3.3 Relating the admissible sets $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$, $\mathcal{M}_c$

One major issue to overcome when studying the relationship between the variational problems (HCP) and (BTPc) for small values of $c \sim 0$, is the fact that the admissible classes for such problems are different in nature: the competitors behave differently near the wells. We address this difficulty by devising procedures that allow us to create curves that are admissible for (HCP) starting from curves that are admissible for (BTPc), and vice versa. Although there are many different ways of achieving this goal, it is crucial to carefully create procedures that enable us to perform an asymptotic analysis as $c \to 0^+$ on the energies $H^c(q_c)$ of the sequence of brake type orbit minimizers $\{q_c\}$ of (BTPc).
Recalling the definitions of $U_{\text{aff}}$ and $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ in (3.10), we see that any $U \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ behaves in such a way that $U(t) \to p_{\pm}$ as $t \to \pm \infty$, meaning that the curve bridges the two components of $\{W = 0\}$:

$$U(t) \text{ approaches } \{W = 0\}^{-} \text{ as } t \to -\infty \text{ and approaches } \{W = 0\}^{+} \text{ as } t \to +\infty.$$  \hfill (3.36)

On the other hand, the admissible class of (BTPc) for $c > 0$ small corresponds to

$$\mathcal{M}_c := \{ q \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) : (i) \ W(q(t)) \geq c \text{ for any } t \in \mathbb{R}, \text{ and } \liminf_{t \to -\infty} \text{dist}(q(t), W^{-}_c) = \liminf_{t \to +\infty} \text{dist}(q(t), W^{+}_c) = 0 \}.$$  \hfill (3.37)

Even though the behavior of an arbitrary curve in $\mathcal{M}_c$ can be rather complicated as $t \to \pm \infty$, we are only interested in a restricted subclass of curves in $\mathcal{M}_c$ that exhibit a simple qualitative behavior at infinity. Such a special curve $U_c : (-\infty, +\infty) \to \mathbb{R}^N$ with $U_c \in \mathcal{M}_c$ behaves so that for some choice of numbers $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$,

$$W(U_c(t)) > c \text{ for every } t \in (t_1, t_2),$$

$$U_c(t) \equiv U_c(t_1) \text{ for all } t \leq t_1, \text{ with } U_c(t_1) \in \{W = c\}^{-},$$

$$U_c(t) \equiv U_c(t_2) \text{ for all } t \geq t_2, \text{ with } U_c(t_2) \in \{W = c\}^{+}.$$  \hfill (3.38)

In other words, the trajectory of such a curve $U_c$ effectively bridges the two components of the $c$-level set $\{W = c\}^{-}$ and $\{W = c\}^{+}$ and it does so in a bounded time interval, and moreover, is asymptotically constant on each end as $t \to \pm \infty$.

### 3.3.1 Constructing curves in $\mathcal{M}_c$ starting from $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$

From the previous discussion about the two admissible classes, it is easy to come up with a procedure so that when a curve $U \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$, admissible for the heteroclinic problem (HCP) is given, then it renders a curve $U|_{\{W \geq c\}} \in \mathcal{M}_c$, that is now admissible for (BTPc) for a small of $c > 0$. In light of (3.36) and (3.38), this association will consist essentially of truncating $U$ to a bounded open interval $I$ where $U \in \{W > c\}$ with endpoints in $\{W = c\}^{\pm}$ and then extending trivially outside $\mathbb{R} \setminus I$. More formally, let us introduce
Definition 3.2 (Truncation of curves $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ at level $c$). Given $c \in (0, c_0)$ and $U \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$, there always exist $-\infty < t_1 < t_2 < +\infty$ in such a way that $U((t_1, t_2)) \subset \{W > c\}$ and $U(t_1) \in \{W = c\}^-$, $U(t_2) \in \{W = c\}^+$. Then the curve

$$U|_{\{W\geq c\}}(t) := \begin{cases} U(t_1) & \text{for } t \in (-\infty, t_1], \\ U(t) & \text{for } t \in (t_1, t_2), \\ U(t_2) & \text{for } t \in [t_2, +\infty). \end{cases}$$

will be called the truncation of $U$ at level $c$.

From this definition of truncation it immediately follows that

$$U|_{\{W\geq c\}} \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N), \quad \inf_{t \in \mathbb{R}} W(U|_{\{W\geq c\}}(t)) \geq c, \quad \liminf_{t \to \pm\infty} \text{dist}(U|_{\{W\geq c\}}(t), W^\pm_c) = 0,$$

so we deduce

Corollary 3.4 (Admissibility of truncation at level $c$). Fix $c \in (0, c_0)$ and $U \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$. Then the truncation of $U$ at level $c$ renders an admissible curve for (BTPc),

$$U|_{\{W\geq c\}} \in \mathcal{M}_c.$$

3.3.2 Constructing curves in $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ starting from $\mathcal{M}_c$

We now continue by introducing another procedure, to create curves in the admissible class $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ of (HCP), provided we dispose with a curve $U_c \in \mathcal{M}_c$ admissible for (BTPc) that in addition satisfies is asymptotically constant in the sense of (3.38). Due to technical reasons it will be necessary to include an auxiliary parameter $\beta$ in the range

$$0 < c < \beta \leq \min\{c_0, c_1\},$$

(3.39)

where $c_0$ is the constant in the properties (W3) through (W4) of $W$, and $c_1$ is the constant in the Lemma 3.1. For any curve $U_c \in \mathcal{M}_c$ there is a choice (not be unique in general) of $t_3, t_4 \in \mathbb{R}$ with $t_3 < t_4$, so that

$$W(U_c(t)) > \beta \quad \text{for every } t \in (t_3, t_4),$$

$$U_c(t_3) \in \{W = \beta\}^-,$$

$$U_c(t_4) \in \{W = \beta\}^+.$$
When this is the case we write

\[ \mathbf{b}^- := U_c(t_3), \quad \mathbf{b}^+ := U_c(t_4). \quad (3.41) \]

Although the dependence on \( c \) and \( \beta \) has been omitted, it will be clear from the context. We would like to extend the portion of the orbit of \( U_c \) lying in \( \{ W > \beta \} \) up to the wells. This new curve, \( U_{c,\beta} : \mathbb{R} \to \mathbb{R}^N \), will be constructed by gluing the restriction \( U_c|_{[t_3,t_4]} \) with the solution of the gradient flow system induced by \( W \) near the wells \( p_\pm \), starting from initial data on the level set \( \{ W = \beta \}^- \) and \( \{ W = \beta \}^+ \), respectively.

**Definition 3.3** (Gradient flow extension of curves in \( \mathcal{M}_c \) from level \( \beta \)). Given \( c \) and \( \beta \) satisfying (3.39), suppose \( U_c \in \mathcal{M}_c \) is admissible for (BTPc) satisfies (3.40) for times \(-\infty < t_3 < t_4 < +\infty\). Then the curve \( U_{c,\beta} : (-\infty, +\infty) \to \mathbb{R}^N \) given below

\[
U_{c,\beta}(t) := \begin{cases} 
\Phi_1(t - t_3, \mathbf{b}^-) & \text{for } t \in (-\infty, t_3], \\
U_c(t) & \text{for } t \in (t_3, t_4), \\
\Phi_2(t - t_4, \mathbf{b}^+) & \text{for } t \in [t_4, +\infty). 
\end{cases}
\]

will be called an extension of \( U_c \) by gradient flow from level \( \beta \). Here \( \mathbf{b}^\pm \) are defined in (3.41) and \( \Phi_1(\cdot, \mathbf{b}^-), \Phi_2(\cdot, \mathbf{b}^+) \) are the solutions to the gradient flow initial value problem (\( GF_1 \))-(\( GF_2 \)), respectively. (see Remark 3.2)

The asymptotic behavior of solutions to the gradient flow system (\( GF_j \)) enables us to construct competitors to the heteroclinic connection problem. More precisely, we have

**Proposition 3.2** (Admissibility of gradient flow extensions). Fix \( c \) and \( \beta \) as in (3.39), and let \( U_c \in \mathcal{M}_c \) be an admissible curve for (BTPc) satisfying (3.40). Then the extension of \( U_c \) by gradient flow from level \( \beta \) renders an admissible curve for (HCP),

\[ U_{c,\beta} \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N). \]

**Proof of Proposition 3.2.** The gradient flow Lemma 3.2 shows that for potentials \( W \) enjoying the non-degeneracy property (W2) at the wells, the approach of \( U_{c,\beta} \) to \( p_- \) (i.e. that of \( \Phi_1(t) \) to \( p_- \)
is exponential as $t \to -\infty$. Similarly, the approach of $U_{c,\beta}$ to $p_+$ (i.e. that of $\Phi_2(t)$ to $p_+$) is exponential as $t \to +\infty$. The admissibility of $U_{c,\beta}$ is immediate.

### 3.3.3 Extensions and restrictions of brake type orbits

Let us recall the fact that, due to the invariance under translations of the functional $H^c$, the “renormalized” sequence $\{q_c\}$ of brake type orbits constructed in (3.21) renders a sequence of minimizers to the variational problems (BTPc). Whence, the value of its corresponding vector-valued Modica-Mortola energy functional at level $c$,

$$H^c(q_c) = \int_{\alpha_c}^{\omega_c} \frac{1}{2} |q_c'|^2 + (W(q_c) - c),$$

with $H^c(q_c) = m_c$, where the latter is $m_0 := \inf\{H^c(q) : q \in \mathcal{M}_c\}$. Moreover, let us recall that the authors in [6] have established that for each value of $c > 0$ there are corresponding finite times $-\infty < \alpha_c < \omega_c < +\infty$ so that the minimizer $q_c \in \mathcal{M}_c$ is trivially constant as $t \to \pm\infty$, in the sense of (3.38) where in this case $t_1 = \alpha_c$ and $t_2 = \omega_c$. Hence, the operation of obtaining a curve $q_{c,\beta}$ as in Definition 3.3, the extension by gradient flow of $q_c$ from level $\beta$ (chosen appropriately) is well-defined.

Let us also note that since $W$ is a double-well potential Corollary 1.2 applies, to yield the existence of a curve $U_0$ minimizing the heteroclinic connection problem (HCP)

$$H^0(U_0) = \int_{-\infty}^{+\infty} \frac{1}{2} |U_0'|^2 + W(U_0),$$

that is, $H^0(U_0) = m_0$, where the latter is $m_0 := \inf\{H^0(U) : U \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)\}$.

The key point in the proof of the main Theorem 3.1 consists of choosing a family of extensions by gradient flow of the brake type orbits from level sets $\beta$, $q_{c,\beta} \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$, so their energies are comparable to that of the aforementioned heteroclinic connection $U_0 \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ for a certain regime in the $\beta$ parameter. Roughly stated, we would like to prove

$$\lim_{c \to 0^+ \atop \beta \to 0^+} |H^0(U_0) - H^0(q_{c,\beta})| = 0.$$  \hspace{1cm} (3.42)
Another ingredient consists of comparing the energies of brake type orbits \( q_c \in \mathcal{M}_c \) with the truncation of the heteroclinic connection \( U_0 \) at level \( c \) (see Definition 3.2):

\[
|H^c(q_c) - H^c(U_0|_{\{W \geq c\}})|.
\]

Interestingly enough, there will be an additional instance where we will need to utilize both constructions simultaneously on a given curve. Indeed, in Lemma 3.7 we will be interested in studying the divergence rate at which \( |T_{c,2c}| \rightarrow +\infty \) as \( c \rightarrow 0^+ \) (time spent by \( q_c \) to bridge \( \{W = 2c\}^- \) with \( \{W = 2c\}^+ \)). This will be accomplished by considering the truncation at level \( c \) of the gradient flow extension of \( q_c \) from level \( c + \sqrt{c} \),

\[
H^c(q_{c,c+\sqrt{c}}|_{\{W \geq c\}}).
\]

We now explicitly apply the extension and truncation techniques developed before to the brake type orbits. The precise time control on the behavior of the brake type orbits \( \{q_c\} \) on level sets of \( W \) (Proposition 3.1 and Corollary 3.2) shows that for any choice of \( c \) and \( \beta \) as in (3.39) the brake type orbit

\[
q_c \in \mathcal{M}_c \text{ satisfies (3.40) at times } t_3 = \tau^-_{c,\beta}, \ t_4 = \tau^+_{c,\beta}, \text{ and these are unique.}
\]

This fact combined with Definition 3.3 yield a unique way of defining the gradient flow extension of the brake type orbit.

**Definition 3.4 (Gradient flow extension of brake type orbits).** Given \( c \) and \( \beta \) satisfying (3.33), the extension by gradient flow from level \( \beta \) of the brake type orbit \( q_c \) corresponds to the following curve

\[
q_{c,\beta}(t) := \begin{cases} 
\Phi_1(t - \tau^-_{c,\beta}, b^-) & \text{for } t \in (-\infty, \tau^-_{c,\beta}], \\
q_c(t) & \text{for } t \in (\tau^-_{c,\beta}, \tau^+_{c,\beta}), \\
\Phi_2(t - \tau^+_{c,\beta}, b^+) & \text{for } t \in [\tau^+_{c,\beta}, +\infty).
\end{cases}
\]

where \( \tau_{c,\beta}^\pm \) are the control times of \( q_c \) on the superlevel set \( \{W > \beta\} \) (see Definition 3.1 and Corollary 3.2), \( b^\pm := q_c(\tau_{c,\beta}^\pm) \), and \( t \mapsto \Phi_j(t,u_0) \) for \( j=1,2 \) is the solution to the gradient flow initial value problem \( (GF_j) \) (see Remark 3.2).
Remark 3.5. It is immediate from the Lemma 3.2 that the gradient flow extension of the brake type orbit \( q_{c,\beta} \), for values of \( c \) and \( \beta \) as in (3.33), satisfies the exponential decay

\[
W(q_{c,\beta}(t)) \leq \begin{cases} 
\beta e^{-\kappa(t-\tau_{c,\beta}^+)} & \text{for any } t \in (\tau_{c,\beta}^+, +\infty), \\
\beta e^{\kappa(t-\tau_{c,\beta}^-)} & \text{for any } t \in (-\infty, \tau_{c,\beta}^-).
\end{cases}
\tag{3.45}
\]

where \( \kappa > 0 \) is the constant in Lemma 3.1. This, combined with the property \((W2^*)\) of the potential, yields the exponential convergence to the wells

\[
|q_{c,\beta}(t) - p_+| \leq \sqrt{\frac{\beta}{\sigma_0}} e^{-\kappa(t-\tau_{c,\beta}^+)/2} \text{ for all } t \in (\tau_{c,\beta}^+, +\infty),
\]

\[
|q_{c,\beta}(t) - p_-| \leq \sqrt{\frac{\beta}{\sigma_0}} e^{\kappa(t-\tau_{c,\beta}^-)/2} \text{ for all } t \in (-\infty, \tau_{c,\beta}^-).
\tag{3.46}
\]

It will be convenient for the next sections to make the following observation.

Corollary 3.5 (Times of gradient flow extensions on superlevel sets of \( W \)). Let \( \epsilon_* \) be the constant in Proposition 3.1. For any choice of

\[
0 < 2c < \beta \leq \epsilon_*,
\tag{3.47}
\]

consider the extension of \( q_c \) by gradient flow from the level \( 2c \). The set of times that \( q_{c,2c} \) spends on the superlevel \( \{W > \beta\} \) can be characterized as

\[
\{ t \in \mathbb{R} : W(q_{c,2c}(t)) > \beta \} = \mathcal{T}_{c,\beta},
\]

where \( \mathcal{T}_{c,\beta} = (\tau_{c,\beta}^-, \tau_{c,\beta}^+) \) is the set of times corresponding to \( q_c \), defined in Corollary 3.2.

Proof of Corollary 3.5. Let us observe from Remark 3.5 that

\[
W(q_{c,2c}(t)) \leq 2ce^{-\kappa|t-\tau_{c,2c}^+|} \leq 2c < \beta \quad \text{for } t \in \mathbb{R} \setminus (\tau_{c,2c}^-, \tau_{c,2c}^+),
\]

which implies \( \{ t \in \mathbb{R} : W(q_{c,2c}(t)) > \beta \} \subset (\tau_{c,2c}^-, \tau_{c,2c}^+) \). Now from the Definition 3.4 of gradient flow extension we have that \( q_{c,2c} = q_c \) on \( (\tau_{c,2c}^-, \tau_{c,2c}^+) \), where in particular \( (\tau_{c,2c}^-, \tau_{c,2c}^+) \supset (\tau_{c,\beta}^-, \tau_{c,\beta}^+) \) in light of Remark 3.4. Then Corollary 3.2 applies, and the proof is now complete. \( \Box \)
We now proceed to describe the truncation at level $c$ of the gradient flow extension of a brake type orbit, see (3.44) for a motivation of such a procedure. Let us observe that the extension by gradient flow of the brake type orbit $q_c$ satisfies

$$q_{c,\beta} \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N),$$

for small parameters $c$ and $\beta$ with $c < \beta$. In particular, there exist distinguished times $T_c^- < 0 < T_c^+$ associated to the solution $\Phi_j$ of the gradient flow $(GF_j)$ starting at $b^\pm \in \{W = \beta\}^\pm$, in such a way that $T_c^-, T_c^+$ are determined by the conditions

$$\Phi_1(T_c^-, b^-) \in \{W = c\}^-, \quad \Phi_2(T_c^+, b^+) \in \{W = c\}^+. \quad (3.48)$$

This definition relies on the fact that for small values of the level set of $W$, there is a unique time at which the gradient flow solution transverses such level set, see assumption (W4). Using (3.48) and Definition 3.4 of gradient flow, we deduce that $q_{c,\beta} \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ satisfies the hypothesis in the procedure of truncation at level $c$ (Definition 3.2) with $t_1 = T_c^- + \tau_{c,\beta}^-$ and $t_2 = T_c^+ + \tau_{c,\beta}^+$. We have,

**Definition 3.5 (Truncation at level $c$ of gradient flow extension).** Consider $c$ and $\beta$ satisfying (3.33). Following Definition 3.2, we say $q_{c,\beta}|_{\{W \geq c\}} : (-\infty, +\infty) \to \mathbb{R}^N$ is the truncation at level $c$ of the extension by gradient flow $q_{c,\beta}$, provided that

$$q_{c,\beta}|_{\{W \geq c\}}(t) := \begin{cases} 
\Phi_1(T_c^-, b^-) & \text{for } t \in (-\infty, T_c^- + \tau_{c,\beta}^-), \\
q_{c,\beta}(t) & \text{for } t \in (T_c^- + \tau_{c,\beta}^-, T_c^+ + \tau_{c,\beta}^+), \\
\Phi_2(T_c^+, b^+) & \text{for } t \in [T_c^+ + \tau_{c,\beta}^+, +\infty),
\end{cases}$$

where $\tau_{c,\beta}^\pm$ are characterized in Corollary 3.2 and $T_c^\pm$ determined by (3.48).

### 3.4 Energy estimates

The main goal of this section is to provide a detailed account of the desired energy analysis (3.42)-(3.43)-(3.44) that is crucial in the proof of the main result Theorem 3.1.
3.4.1 Preliminary energy estimates

Our analysis starts with an estimate of the energy, at level \( c \), of the truncation at level \( c \) of the gradient flow extension of \( q_c \).

**Lemma 3.4 (Energy Bound I).** Let \( c \) and \( \beta \) satisfy condition (3.33). Then

\[
H^c(q_{c,\beta}|_{W \geq c}) \leq H^c_{\Xi_{c,\beta}}(q_c) + \frac{(\kappa + 2)}{\kappa} \beta,
\]

where \( \kappa, \varpi \) are as in Lemma 3.1, and \( \Xi_{c,\beta} \) is defined in (3.34). In particular,

\[
H^c(q_{c,\beta}|_{W \geq c}) \leq m_c + O(\beta),
\]

with \( m_c \) the value of the infimum in (BTPc).

**Proof of Lemma 3.4.** Using the Definition 3.5 of the truncation we observe that

\[
H^c(q_{c,\beta}|_{W \geq c}) = \left\{ \int_{\tau_{c,-}^\beta}^{\tau_{c,+}^\beta} + \int_{\tau_{c,+}^\beta}^{\tau_{c,+}^\beta + T_c^+} \right\} \left( \frac{1}{2} |q_{c,\beta}'|^2 + W(q_{c,\beta}) - c \right) dt
\]

\[=: I_- + H^c_{(\tau_{c,-}^\beta,\tau_{c,+}^\beta)}(q_c) + I_+.
\]

It only suffices to prove a bound for \( I_+ \) since both bounds, including the one of \( I_- \), rely on estimations on the solutions \( \Phi_j \) for \( j = 1, 2 \) of \((GF_j)\), which behave in the same fashion. Performing the change of variables \( t \mapsto t - \tau_{c,+}^\beta \) on the integral \( I_+ \), we readily see the definition of \( q_{c,\beta} \) that

\[
I_+ = \int_0^{T_c^+} \frac{1}{2} |\Phi_2'(t, b^+)|^2 + (W(\Phi_2(t, b^+)) - c) dt
\]

\[= \int_0^{+\infty} \frac{1}{2} |\Phi_2'(t, b^+)|^2 + W(\Phi_2(t, b^+))dt - c|T_c^+|
\]

\[\leq \frac{(\kappa + 2)}{2\kappa} \beta,
\]

where in the last step follows from Corollary 4.3, using that \( W(b^+) = \beta \). \( \square \)

The second step in our analysis, is to establish analogous bounds for the energy at level zero, \( H^0 \), of the gradient flow extension \( q_{c,\beta} \) when considered a competitor in (HCP).
Lemma 3.5 (Energy bound II). Let \( c \) and \( \beta \) satisfy condition (3.33). Then

\[
H^0(q_{c,\beta}) \leq H^c_{\Sigma_{c,\beta}}(q_c) + \frac{(\kappa + 2)}{\kappa} \beta + c|\Sigma_{c,\beta}|,
\]

where \( \kappa, \kappa \) are the constant as in Lemma 3.1, and \( \Sigma_{c,\beta} \) is defined in (3.34). In particular,

\[
H^0(q_{c,\beta}) \leq m_c + O(\beta) + c|\Sigma_{c,\beta}|,
\]

with \( m_c \) the value of the infimum in (BTPc).

Proof of Lemma 3.5. Let us observe from the Definition 3.4 of gradient flow extension that

\[
H^0(q_{c,\beta}) = \int_{-\infty}^{+\infty} \frac{1}{2} |q'_{c,\beta}|^2 + W(q_{c,\beta}) \, dt = \left( \int_{-\infty}^{\tau_{c,\beta}^-} + \int_{\tau_{c,\beta}^-}^{\tau_{c,\beta}^+} + \int_{\tau_{c,\beta}^+}^{+\infty} \right) \frac{1}{2} |q'_{c,\beta}|^2 + W(q_{c,\beta}) \, dt
\]

\[
= \hat{I}_- + (H_{\Sigma_{c,\beta}}^{(\tau_{c,\beta}^-, \tau_{c,\beta}^+)}(q_c) + c|\tau_{c,\beta}^+ - \tau_{c,\beta}^-|) + \hat{I}_+.
\]

The arguments in the proof of Energy bound I, Lemma 3.4, can be applied to establish the bound

\[
\hat{I}_- + \hat{I}_+ \leq (\kappa + 2)\beta/\kappa,
\]

resulting from the exponential convergence of \( q_{c,\beta} \) to the wells (Lemma 3.2).

\[
\square
\]

3.4.2 A finer energy analysis

We would like now to control the time spent by the brake-type orbit \( q_c \) on the superlevel set \( \{ W > 2c \} \), as \( c \to 0^+ \). As it turns out, such a control is necessary in order to justify, in various senses, the convergence of the variational problems for brake type orbits to the heteroclinic variational problem, at level zero.

We start by finding a bound on the optimal values of the sequence of problems (BTPc), uniformly in the parameter \( c > 0 \).

Lemma 3.6. Consider \( c_0 > 0 \) as in assumptions (W3) through (W4). Then

\[
\sup_{c \in (0, c_0)} m_c \leq |p_+ - p_-| \max_{[p_-, p_+]} \sqrt{2W},
\]

where \([p_-, p_+]\) denotes the line segment in \( \mathbb{R}^N \) joining \( p_- \) to \( p_+ \).
Proof of Lemma 3.6. Given $\varepsilon > 0$, consider the parametrization of the line segment joining the
two wells, $U^\varepsilon_{\text{aff}}(t) = (2\varepsilon)^{-1}[(\varepsilon - t)\mathbf{p}_- + (\varepsilon + t)\mathbf{p}_+]$ for $t \in [-\varepsilon, \varepsilon]$. Fixing $c \in (0, c_0)$ and $\varepsilon > 0$ we
now construct a restriction of $U^\varepsilon_{\text{aff}}$ to \{W \geq c\}. First define $t_c^- := \max\{t \in [-1,1] : U^\varepsilon_{\text{aff}}(t) \in W_c^{-}\}$,
$t_c^+ := \min\{t \in [-1,1] : U^\varepsilon_{\text{aff}}(t) \in W_c^{+}\}$ and set
\[
U^{c,\varepsilon}_{\text{aff}}(t) = \begin{cases} 
U^\varepsilon_{\text{aff}}(t_c^-) & \text{for } t \in (-\infty, t_c^-], \\
U^\varepsilon_{\text{aff}}(t) & \text{for } t \in (t_c^-, t_c^+), \\
U^\varepsilon_{\text{aff}}(t_c^+) & \text{for } t \in [t_c^+, +\infty). 
\end{cases}
\]
It readily follows that $U^{c,\varepsilon}_{\text{aff}} \in \mathcal{M}_c$ and so the minimizing character of $q_c$ implies
\[
m_c = H^c(q_c) \leq H^c(U^{c,\varepsilon}_{\text{aff}}) = \int_{t_c^-}^{t_c^+} \frac{1}{2} |(U^{c,\varepsilon}_{\text{aff}})'|^2 + (W(U^{c,\varepsilon}_{\text{aff}}) - c) \, dt \\
\leq \int_{-\varepsilon}^{\varepsilon} \frac{1}{2} \left| \frac{\mathbf{p}_+ - \mathbf{p}_-}{2\varepsilon} \right|^2 + W \left( \frac{\varepsilon - t}{2\varepsilon} \right) \mathbf{p}_- + \left( \frac{\varepsilon + t}{2\varepsilon} \right) \mathbf{p}_+ \right) \, dt \\
\leq \frac{1}{4\varepsilon} |\mathbf{p}_+ - \mathbf{p}_-|^2 + 2\varepsilon \max_{|\mathbf{p}_- - \mathbf{p}_+|} W =: f(\varepsilon),
\]
independently of $c \in (0, c_0)$. The least upper bound is found by minimizing $f(\varepsilon)$ on $(0, \infty)$.

Lemma 3.7 (Crossing times lemma). Consider the collection of intervals \{\mathcal{I}_{c,2c}\} defined
by (3.34) associated to the minimizers \{q_c\} of the family of problems \{(BTPc)\}, for $c$ ranging
in $(0, c_*/2)$, where $c_*$ is as in Corollary 3.2. Then the following assertions hold
\[
\limsup_{c \to 0^+} |\mathcal{I}_{c,2c}| = +\infty. \tag{3.49}
\]
Furthermore,
\[
\limsup_{c \to 0^+} c |\mathcal{I}_{c,2c}| = 0. \tag{3.50}
\]
Remark 3.6. In view of Corollary 3.2, the two limits above show that the “amount of time” spent
by $q_c$ on the $2c$-superlevel set of $W$, namely $|\mathcal{I}_{c,2c}| = |\tau_{c,2c}^+ - \tau_{c,2c}^-|$, diverges as $c \to 0^+$ and moreover
it does so at a rate strictly slower than $1/c$. In particular, it is worth noting that the period of the
brake type orbit $q_c$ diverges as $c \to 0^+$, due to (3.49). Namely,
\[
\limsup_{c \to 0^+} |\omega_c - \alpha_c| = +\infty.
\]
Proof of Lemma 3.7. We argue (3.49) by contradiction assuming \( \limsup_{c \to 0^+} |\tau_{c,2c}^+ - \tau_{c,2c}^-| < +\infty \), and let us ease the notation by writing \( l_{2c} := |\tau_{c,2c}^+ - \tau_{c,2c}^-| \) for \( c > 0 \). With no loss of generality assume that \( l_{2c_n} \to l_* \in \mathbb{R} \) as \( c_n \to 0^+ \), up to a subsequence. Let us break the analysis in two possible scenarios,

(i) \( l_* = 0 \).

(ii) \( \delta \leq l_* \leq \delta^{-1} \), for some \( \delta \in (0,1] \).

First we establish (i) cannot occur. Indeed, if \( l_{2c_n} \to 0^+ \) then by considering the shifted sequence \( \tilde{q}_n(t) := q_{c_n}(t + \tau_{c_n,2c_n}^-) \) we readily see

\[
\int_0^{l_{2c_n}} |\tilde{q}_n(t)| dt \geq \frac{1}{l_{2c_n}}|\tilde{q}_n(l_{2c_n}) - \tilde{q}_n(0)| = \frac{1}{l_{2c_n}}|q_{c_n}(\tau_{c_n,2c_n}^+) - q_{c_n}(\tau_{c_n,2c_n}^-)| \geq \frac{1}{2l_{2c_n}}|\mathbf{p}_+ - \mathbf{p}_-| \to +\infty,
\]

where the last inequality follows from the comparison Lemma 3.1-(3.12), provided \( n > n_0 \) is chosen large enough so \( c_n \) approaches 0. This fact along with the continuity of \( \tilde{q}_n \) yields \( |\tilde{q}_n(0)| \geq 1 \), i.e. \( |q_{c_n}(\tau_{c_n,2c_n}^-)| \geq 1 \), if \( n > n_1 \geq n_0 \) is taken even larger. However, the equipartition of energy relation (3.3) yields a contradiction, since \( |q_{c_n}^\prime(\tau_{c_n,2c_n}^-)|^2 = 2c_n \to 0^+ \). Now we treat case (ii).

Normalize the sequence by letting

\[
U_n : [0,1] \to \mathbb{R}^N, \quad U_n(s) := q_{c_n}(l_{2c_n} s + \tau_{c_n,2c_n}^-) \quad \text{for} \ s \in [0,1].
\]

From the properties enjoyed by \( q_c \), it follows that, each \( U_n \) is a classical solution to

\[
U_n^{\prime\prime}(s) = \frac{l_{2c_n}^2}{2} \nabla W(U_n(s)) \quad \text{for} \ s \in [0,1],
\]

and furthermore, it satisfies a pointwise equipartition property

\[
\frac{1}{2} |U_n^\prime(s)|^2 = \frac{l_{2c_n}^2}{2} (W(U_n(s)) - c_n) \quad \text{for} \ s \in [0,1].
\]

Here the derivatives of \( U_n \) at the endpoints are computed using the corresponding one-sided limit \( 0^+ \) and \( 1^- \). Let us recall from the behavior (W1) of the potential at infinity that the minimizers
$q_{cn}$ satisfy an $L^\infty$-bound, which combined with (3.51)-(3.52) and $l_\ast \leq \delta^{-1}$, in turn imply

\[
\sup_n \|U_n\|_{L^\infty([0,1];\mathbb{R}^N)} \leq R_0,
\]

\[
\sup_n \|U_n\|_{L^\infty([0,1];\mathbb{R}^N)} \leq \sqrt{2}(1 + \delta^{-1}) \max_{|u| \leq R_0} W(u),
\]

\[
\sup_n \|U_n\|_{L^\infty([0,1];\mathbb{R}^N)} \leq (1 + \delta^{-1})^2 \max_{|u| \leq R_0} |\nabla W(u)|.
\]

Then by Arzela-Ascoli, there exists a subsequence and some $U_\ast \in C^1([0,1];\mathbb{R}^N)$ up to which

\[
U_n \xrightarrow{n \to \infty} U_\ast \quad \text{in} \quad C^1([0,1];\mathbb{R}^N).
\]

Now, since $U_n$ is in particular a weak solution to (3.51) and converges to $U_\ast$, then a passage to the limit as $n \to \infty$ in (3.51) reveals that $U_\ast$ is a weak (hence classical) solution to

\[
U'' = l_\ast^2 \nabla W(U) \quad \text{in} \quad [0,1].
\]

Moreover, the $C^1$ convergence yields in (3.52)

\[
\frac{1}{2} |U_n'|^2 = l_\ast^2 W(U_\ast) \quad \text{in} \quad [0,1],
\]

\[
U_\ast(0) = \lim_{n \to \infty} q_{cn}(\tau_{cn,2cn}^-) = p_-, \quad U_\ast(1) = \lim_{n \to \infty} q_{cn}(\tau_{cn,2cn}^+) = p_+.
\]

Nonetheless, the uniqueness of solutions to the Cauchy problem

\[
q'' = l_\ast^2 \nabla W(q),
\]

\[
q(0) = p_-, \quad q'(0) = 0,
\]

shows that $U_\ast(s) \equiv p_-$, thus contradicting the fact that $U_\ast$ connects $p_-$ to $p_+$. This shows that is also impossible, thus finishing the proof of (3.49).

Now we continue by proving (3.50). Recall from Corollary 3.3 that the sets of times $\mathcal{T}_{c,\beta}$ for different values of $\beta$ and $c$ fixed, where $W(q_c(t)) > \beta$, are strictly contained in one another. Taking $0 < c << 1$ small enough so that $2c < c + \sqrt{c} < c_\ast$, we split the larger interval as

\[
\mathcal{T}_{c,2c} = \mathcal{T}_{c,c+\sqrt{c}} \cup I_c, \quad \text{where}
\]

\[
I_c := \mathcal{T}_{c,2c} \setminus \mathcal{T}_{c,c+\sqrt{c}}.
\]
From (3.34)-(3.35) we readily see $2c < W(q_c(t)) \leq c + \sqrt{c}$ for all $t \in I_c$, then

$$H^c_{I_c}(q_c) \geq \int_{I_c} (W(q_c) - c) \, dt > c|I_c|.$$ 

Using the fact that $q_c$ is a minimizer for (BTPc), we compare its energy with the one of the truncation at level $c$ of the gradient flow extension from level $c + \sqrt{c}$. That is,

$$H_{\xi,c;2c}^c(q_c) \leq H^c(q_c) \leq H^c(q_{c,c+\sqrt{c}}|\{W \geq c\}).$$

Applying the Energy bound I (Lemma 3.4) with $\beta = c + \sqrt{c}$, and recalling (3.53) we deduce

$$H_{\xi,c+\sqrt{c}}^c(q_c) + H^c_{I_c}(q_c) \leq H_{\xi,c+\sqrt{c}}^c(q_c) + \frac{\kappa + 2}{\kappa} (c + \sqrt{c}),$$

whence

$$\limsup_{c \to 0} 2c |I_c| \leq \limsup_{c \to 0} O(c + \sqrt{c}) = 0.$$ 

On the other hand, one easily gets

$$m_c = H^c(q_c) \geq \int_{\xi,c+\sqrt{c}} (W(q_c) - c) \, dt \geq \sqrt{c}|\xi_{c,c+\sqrt{c}}|.$$ 

Thus

$$\limsup_{c \to 0} c |\xi_{c,c+\sqrt{c}}| \leq \limsup_{c \to 0^+} \sqrt{c} m_c = 0,$$

since $\sup_{c \in (0,c_0)} m_c < \infty$ by Lemma 3.6.

A key consequence of the asymptotic behavior of the crossing times on superlevel sets $\{W > 2c\}$, is the following sharp energy estimation:

**Proposition 3.3 (Energy Bound III).** For any $0 < 2c < c^*$, where $c^*$ is the constant in Corollary 3.2, one has

$$H^0(q_{c,2c}) \leq m_c + o_c(1),$$

where $\lim_{c \to 0^+} o_c(1) = 0$. 

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Proof of Proposition 3.3. The Energy Bound II (Lemma 3.5) shows that for \( \beta = 2c \),

\[
H^0(q_c,2c) \leq m_c + \frac{\kappa + 2}{\kappa} 2c + c|\Sigma_{c,2c}|
= m_c + o_c(1),
\]

where the last identity follows directly from the Crossing Times Lemma 3.7. \( \square \)

Remark 3.7. Lemma 3.7 yields the existence of a small fixed constant \( c' \in (0, c_*) \) so that

\[
o_c(1) = \frac{\kappa + 2}{\kappa} c' + c'|\Sigma_{c',2c'}| < |p_+ - p_-| \max_{[p_-,p_+]} \sqrt{2W}.
\]

By shrinking the value of \( c_* \) in Proposition 3.1 if necessary, we assume with no loss of generality that \( c' = c_* \). In particular, for this choice of \( c_* \) we have

\[
\sup_{c \in (0,c_*/2)} H^0(q_c,2c) \leq M_0 := 2|p_+ - p_-| \max_{[p_-,p_+]} \sqrt{2W}, \tag{3.54}
\]

in light of Proposition 3.3.

3.5 Analysis of optimal values \( \{m_c\} \) near \( c = 0 \)

The above analysis suggests that the convergence of \( \{q_c,2c\} \) as \( c \to 0^+ \) to a heteroclinic connection minimizing (HCP), might depend upon the asymptotic behavior of the sequence of optimal values \( m_c \) for (BTPc). If this convergence is to be true, one can expect that the limiting values of \( m_c \) as \( c \to 0^+ \) match \( m_0 \), the latter defined as the optimal value of (HCP).

We start by observing that \( \{m_c\} \) is an increasing sequence as \( c \to 0^+ \), so Lemma 3.6 then ensures this sequence is convergent. More precisely, we have

Lemma 3.8 (Monotonicity of optimal values). The sequence of optimal values \( \{m_c\} \) of the family of variational problems \( \{(BTPc)\} \) satisfies

\[
m_c > m_{c'} + (c' - c)|\Sigma_{c,c'}| \quad \text{for} \quad 0 < c < c' \leq c_*,
\]

where \( c_* \) is the constant in Corollary 3.2, and \( \Sigma_{c,c'} \) is defined in (3.34).
Proof of Lemma 3.8. Consider the minimizer $q_c$ of (BTPc), so $m_c = H_{(\alpha_c, \omega_c)}(q_c)$. Recalling from Corollaries 3.2-3.3 the definition and inclusion properties of $\beta \mapsto \mathcal{T}_{c, \beta}$ for $\beta = c' > c$, we deduce $(\alpha_c, \omega_c) = \lim_{\beta \to +} \mathcal{T}_{c, \beta} \supset \mathcal{T}_{c, c'}$, and therefore

$H_{(\alpha_c, \omega_c)}(q_c) > H_{\mathcal{T}_{c, c'}}(q_c) = \int_{\mathcal{T}_{c, c'}} \frac{1}{2} |q'_c|^2 + (W(q_c) - c) \, dt$

$= \int_{\mathcal{T}_{c, c'}} \left( \frac{1}{2} |q'_c|^2 + W(q_c) - c' \right) \, dt + (c' - c) |\mathcal{T}_{c, c'}|$

$= H_{\mathcal{T}_{c, c'}}(q_c) + (c' - c) |\mathcal{T}_{c, c'}|.$

If we denote by $\hat{q}_{c, c'}$ the trivial extension of $q_c|_{\mathcal{T}_{c, c'}}$, namely, $\hat{q}_{c, c'}(t) \equiv q_c(\tau_{c, c'}^{-})$ for all $t \leq \tau_{c, c}'^{-}$, and $\hat{q}_{c, c'}(t) \equiv q_c(\tau_{c, c'}^{+})$ for all $t \geq \tau_{c, c}'^{+}$, then it clearly follows that $\hat{q}_{c, c'} \in \mathcal{M}_{c'}$ and also $H_{\mathcal{T}_{c, c'}}(q_c) = H_{c'}(\hat{q}_{c, c'})$. Consequently the definition of the infimum $m_{c'}$ in (BTPc') yields

$H_{(\alpha_c, \omega_c)}(q_c) > H_{c'}(\hat{q}_{c, c'}) + (c' - c) |\mathcal{T}_{c, c'}|$

$\geq m_{c'} + (c' - c) |\mathcal{T}_{c, c'}|.$

The next step is to highlight one of the essential features about the relationship between the variational problems (BTPc) and (HCP), namely, the convergence of the optimal values for the family of variational problems, as $c \to 0^+$.

Theorem 3.2 (Asymptotics of Optimal Values). For a potential $W : \mathbb{R}^N \to \mathbb{R}$ satisfying (W1) through (W4), the optimal values \{m_c\} of the family of variational problems \{(BTPc)\} satisfy

$$\lim_{c \to 0^+} m_c = m_0,$$

where $m_0$ is the optimal value of the heteroclinic variational problem (HCP).

Proof of Theorem 3.2. Given $U_0$ a heteroclinic connection, minimizer to (HCP), consider $\epsilon_*$ as in Corollary 3.2. There exists $\tilde{t} \in \mathbb{R}$ so that $W(U(\tilde{t})) > \epsilon_*$, then for $c \in (0, \epsilon_*)$ define

$\hat{\tau}_{0, c}^+ := \sup \{ t > \tilde{t} : \min_{[\tilde{t}, t]} W(U_0) > c \},$

$\hat{\tau}_{0, c}^- := \inf \{ t < \tilde{t} : \min_{[t, \tilde{t}]} W(U_0) > c \}.$
This construction guarantees that
\[ W(U_0(t)) > c \text{ for } t \in (\tau_{0,c}^-, \tau_{0,c}^+), \quad \text{and} \quad U_0(\tau_{0,c}^+) \in \partial\{W \leq c\}^+. \]

For these times we consider \( U_0|_{\{W \geq c\}} \), the truncation at level \( c \) of \( U_0 \in H^1_{\text{aff}}(\mathbb{R}^d; \mathbb{R}^N) \) as given in Definition 3.2 with \( t_1 = \tau_{0,c}^- \) and \( t_2 = \tau_{0,c}^+ \). Observe \( U_0|_{\{W \geq c\}} \in \mathcal{M}_c \) is admissible for (BTP\( c \)) by virtue of Corollary 3.4, whence

\[
m_c \leq H^0(U_0|_{\{W \geq c\}}) = \int_{\tau_{0,c}^-}^{\tau_{0,c}^+} \frac{1}{2} |U'|^2 + (W(U_0) - c) \, dt
\]
\[
= H^0(U_0|_{\tau_{0,c}^-, \tau_{0,c}^+})(U_0) - c|\tau_{0,c}^+ - \tau_{0,c}^-|
\]
\[
\leq H^0(U_0) = m_0. \tag{3.55}
\]

This proves \( \limsup_{c \to 0^+} m_c \leq m_0 \). Conversely, for any heteroclinic minimizer \( U_0 \) of (HCP), Proposition 3.2 allows us to compare \( m_0 \) with the energy of the gradient flow extension of \( q_c \) from level \( 2c \). The Energy Bound III (Proposition 3.3) shows this energy is comparable to that of \( q_c \) as well:

\[
m_0 = H^0(U_0) \leq H^0(q_c, 2c) \leq m_c + o_c(1). \tag{3.56}
\]

This yields \( m_0 \leq \limsup_{c \to 0^+} m_c \). Finally, we observe the limit of \( m_c \) exists due to Lemma 3.8. \( \square \)

An immediate consequence of (3.55) and (3.56) is

**Corollary 3.6.** The sequence \( \{q_c, 2c\} \) of gradient flow extensions of \( q_c \) at level \( 2c \), is a minimizing sequence for (HCP):

\[
\lim_{c \to 0} H^0(q_c, 2c) = m_0.
\]

### 3.6 Proof of the main Theorem 3.1

The desired convergence of the brake-type orbits \( \{q_c\} \) (more precisely that of \( \{q_c, 2c\} \)) to a heteroclinic connection between the wells will be a consequence of a delicate control on the times where the sequence spends on superlevel sets of the potential \( W \), uniformly on the parameter \( c \) when is close to 0. The energy analysis performed in the previous section will show key to this purpose.
Lemma 3.9 (Time control gradient flow extension on sublevel sets of $W$). Let $c_*, M_0$ be the constants in Remark 3.7. Then for any choice of $\beta \leq c_*$ we have

$$W(q_{c,2c}(t)) \leq \beta \quad \text{for all} \quad |t| \geq M_0/\beta,$$

uniformly on $c \in (0, \beta/2)$.

Proof of Lemma 3.9. Let us recall from Corollary 3.5 that $W(q_{c,2c}(t)) > \beta$ precisely for $t \in \Xi_{c,\beta}$, where $\Xi_{c,\beta}$ is defined in Corollary 3.2. In this way, Remark 3.7 shows that for a choice of parameters $0 < 2c < \beta \leq c_*$, it follows that

$$M_0 \geq H^0(q_{c,2c}) \geq \int_{\Xi_{c,\beta}} W(q_{c,2c}(t)) \, dt > \beta |\Xi_{c,\beta}|.$$

On the other hand, from Corollary 3.2 we also know $\Xi_{c,\beta}$ is connected and $0 \in (\tau^-_{c,\beta}, \tau^+_{c,\beta}) = \Xi_{c,\beta}$. The above bound then ensures

$$\Xi_{c,\beta} \subset (-M_0/\beta, M_0/\beta),$$

which finishes the proof of this lemma.

In the next step we prove the key feature of the sequence $\{q_{c,2c}\}$ that will render the desired convergence to a heteroclinic connection, namely, the boundedness of $\{q_{c,2c}\}$ in the $H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ topology.

Proposition 3.4 (Uniform estimates of gradient flow extensions). Let $c_*, M_0$ be the constant in Remark 3.7, $\sigma_0$ the constant in property (W2*) of $W$, and $U_{\text{aff}}$ the affine map joining the wells in (3.10). Then,

$$\sup_{c \in (0,c_*/2)} \|q_{c,2c}\|_{C^{0,1/2}(\mathbb{R}; \mathbb{R}^N)} \leq \sqrt{2M_0 + \max_{\{u: W(u) \leq c_*\}} |u| + \max \left\{ \frac{c_*}{\sqrt{\sigma_0}}, M_0 \sqrt{2} \right\}}, \quad (3.57)$$

$$\sup_{c \in (0,c_*/2)} \|q'_{c,2c} - U'_{\text{aff}}\|_{L^2(\mathbb{R}; \mathbb{R}^N)} \leq \sqrt{4M_0 + |\mathbf{p}_+ - \mathbf{p}_-|^2}. \quad (3.58)$$
Moreover, setting \( \beta_0 := \min\{M_0/2, \epsilon_*\} \), there also holds

\[
\sup_{c \in (0, \beta_0/2)} \| q_{c,2c} - U_{\text{aff}} \|_{L^2(\mathbb{R}; \mathbb{R}^N)} \leq \left\{ \frac{4M_0}{\beta_0} \left( \sup_{c \in (0, \epsilon_*/2)} \| q_{c,2c} \|_{L^\infty(\mathbb{R}; \mathbb{R}^N)} + \max\{ |p_+|^2, |p_-|^2 \} \right) + \frac{M_0}{\sigma_0} \right\}^{1/2}.
\] (3.59)

Proof of Proposition 3.4. To argue the first bound (3.57), let us first observe that there is a continuous embedding \( H^1(\mathbb{R}; \mathbb{R}^N) \hookrightarrow C^{0,1/2}(\mathbb{R}; \mathbb{R}^N) \): for any \( t' < t'' \), the Cauchy-Schwarz inequality yields

\[
|q_{c,2c}(t'') - q_{c,2c}(t')| \leq \int_{t'}^{t''} |q'_{c,2c}(t)| \leq \sqrt{t'' - t'} \left( \int_{t'}^{t''} |q'_{c,2c}(t)|^2 dt \right)^{1/2} \leq \sqrt{2M_0(t'' - t')},
\] (3.60)
uniformly in \( c \in (0, \epsilon_*/2) \) due to Remark 3.7 (see (3.54)). This yields an upper bound in the 1/2-Hölder seminorm

\[
\sup_{c \in (0, \epsilon_*/2)} |q_{c,2c}|_{C^{0,1/2}(\mathbb{R}; \mathbb{R}^N)} \leq \sqrt{2M_0}.
\] (3.61)

We also recall from Corollary 3.5 that the set of times in \( \mathbb{R} \) where the gradient flow extensions \( q_{c,2c} \) lies on the superlevel set \( \{W > \epsilon_*\} \) is precisely \( \Xi_{c,\epsilon_*} = (\tau_{c,\epsilon_*}, \tau_{c,\epsilon_*}^+) \). In particular, taking any \( t' \in \Xi_{c,\epsilon_*} \) and \( t'' = \tau_{c,\epsilon_*}^+ \) in the inequality (3.60), we obtain

\[
|q_{c,2c}(t')| \leq |q_c(\tau_{c,\epsilon_*}^+)| + \sqrt{2M_0|\Xi_{c,\epsilon_*}|},
\] (3.62)
from which the trivial bound follows \( |q_c(\tau_{c,\epsilon_*}^+)| \leq \max\{u; W(u) = \epsilon_*\} |u| \). Also, from the monotonicity of optimal values Lemma 3.8 (together with Lemma 3.6) we see that

\[
\sup_{c \in (0, \epsilon_*/2)} |\Xi_{c,\epsilon_*}| \leq \frac{2}{\epsilon_*} \left( \sup_{c \in (0, \epsilon_*/2)} m_c - m_{\epsilon_*} \right) \leq \frac{M_0}{\epsilon_*}.
\]

All in all, we deduce from (3.62) that

\[
\sup_{t' \in \Xi_{c,\epsilon_*}} |q_{c,2c}(t')| \leq \max\{u; W(u) = \epsilon_*\} |u| + \frac{\sqrt{2M_0}}{\sqrt{\epsilon_*}} \quad \text{for all} \ c \in (0, \epsilon_*/2).
\] (3.63)

Additionally, given that \( W(q_{c,2c}(t)) \leq \epsilon_* \) for every \( t \notin \Xi_{c,\epsilon_*} \), property (W2*) implies that \( \sigma_0 |q_{c,2c}(t) - p_\pm|^2 \leq \epsilon_* \) for such \( t \), thus giving

\[
|q_{c,2c}(t)| \leq |p_-| + \sqrt{\epsilon_*/\sigma_0} \quad \text{for} \ t < \tau_{c,\epsilon_*}^-,
\]
\[
|q_{c,2c}(t)| \leq |p_+| + \sqrt{\epsilon_*/\sigma_0} \quad \text{for} \ t > \tau_{c,\epsilon_*}^+.
\]
In other words,
\[
\sup_{t \notin \mathbb{T}_{c,t_*}} |q_{c,2c}(t)| \leq \max\{|p_0|, |p_+|\} + \sqrt{c_*/\sigma_0}. \tag{3.64}
\]

Then the \(C^0\)-bound follows from (3.63)-(3.64) combined
\[
\sup_{c \in (0,c_*/2)} \|q_{c,2c}\|_{C^0(\mathbb{R};\mathbb{R}^N)} \leq \max\{u:W(u) \leq c_*\} \|u\| + \max\left\{\sqrt{c_*}, M_0\sqrt{\frac{2}{c_*}}\right\}, \tag{3.65}
\]

Therefore, the first bound follows from (3.61)-(3.65) by recalling the definition of the norm
\[
\|q_{c,2c}\|_{C^0,1/2(\mathbb{R};\mathbb{R}^N)} := [q_{c,2c}]_{C^0,1/2(\mathbb{R};\mathbb{R}^N)} + \|q_{c,2c}\|_{C(\mathbb{R};\mathbb{R}^N)}.
\]

The second bound (3.58) follows directly from Remark 3.7-(3.54) and the definition of the affine map \(U_{aff}\); for all \(c \in (0,c_*/2)\),
\[
\int_{-\infty}^{+\infty} |q'_{c,2c} - U'_{aff}|^2 \leq 2 \int_{-\infty}^{+\infty} (|q'_{c,2c}|^2 + |U'_{aff}|^2)
\leq 4H^0(q_{c,2c}) + 2 \int_{-1}^{1} |U'_{aff}|^2 \leq 4M_0 + |p_+ - p_-|^2.
\]

Now to argue the third bound (3.59), first we note that Lemma 3.9 yields in light of \(\beta_0 \leq c_*\), that \(W(q_{c,2c}(t)) \leq \beta_0\) for every \(|t| \geq M_0/\beta_0\), uniformly in \(c \in (0,\beta_0/2)\). In addition, we can apply the property \((W^2)\) of \(W\) in light of the inequality \(c_* \leq c_0\), to get
\[
|q_{c,2c}(t) - p_{\pm}|^2 \leq \frac{1}{\sigma_0} W(q_{c,2c}(t)) \quad \text{for all} \quad |t| \geq \frac{M_0}{\beta_0}, \tag{3.66}
\]
uniformly for \(c \in (0,\beta_0/2)\). But then the uniform bound on the energies (3.54) of \(\{q_{c,2c}\}\) together with (3.66) yields,
\[
M_0 \geq H^0(q_{c,2c}(t)) \geq \left(\int_{M_0/\beta_0}^{+\infty} + \int_{-\infty}^{-M_0/\beta_0}\right) W(q_{c,2c}(t)) \ dt
\geq \int_{M_0/\beta_0}^{+\infty} \sigma_0 |q_{c,2c}(t) - p_{\pm}|^2 \ dt + \int_{-\infty}^{-M_0/\beta_0} \sigma_0 |q_{c,2c}(t) - p_{\pm}|^2 \ dt,
\]
independently of \(0 < c < \beta_0/2\). On the other hand, for \(\beta_0\) chosen so \(M_0/\beta_0 > 1\), we have \(U_{aff} \equiv p_-\) on \((-\infty,-M_0/\beta_0]\) and \(U_{aff} \equiv p_+\) on \([M_0/\beta_0, +\infty)\). Combining this fact with the above inequality, we conclude
\[
\|q_{c,2c} - U_{aff}\|_{L^2(I;\mathbb{R}^N)}^2 \leq \frac{M_0}{\sigma_0} \quad \text{for} \quad I := \mathbb{R} \setminus \left(-\frac{M_0}{\beta_0}, \frac{M_0}{\beta_0}\right).
\]

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Finally we observe
\[ \int_{-M_0/\beta_0}^{M_0/\beta_0} |q_{c,2c}(t) - U_{\text{aff}}(t)|^2 \, dt \leq \frac{4M_0}{\beta_0} \left( \|q_{c,2c}\|_{L^\infty(\mathbb{R};\mathbb{R}^N)}^2 + \max\{|p^-|^2, |p^+|^2\} \right), \]
where the last term of the inequality comes from \( \|U_{\text{aff}}\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} = \max\{|p^-|, |p^+|\} \). Thus, the desired bound (3.59) is a consequence of the last two inequalities combined.

We are now in position to give a full proof of the main theorem of Chapter 3, which for the sake of completeness, we restate below:

**Theorem 3.1.** Assume \( W \) satisfies (W1) through (W4). Then for any sequence \( c_n \to 0^+ \), the family of variational problems \( \{\text{BTP}_{c_n}\} \) approaches the variational heteroclinic connection problem (HCP), in the following senses:

For any sequence \( \{q_{c_n}\} \) of minimizers to (BTP\(_{c_n}\)), there holds

(i) \( \lim_{n \to \infty} m_{c_n} = m_0 \), where \( m_{c_n} := H^{c_n}(q_{c_n}) \) and \( m_0 := \inf\{H^0(V) : V \in H^1_{\text{aff}}(\mathbb{R};\mathbb{R}^N)\} \).

(ii) There exist times \( \{t_k\} \subset \mathbb{R} \), and a subsequence \( \{q_{c_{n_k},2c_{n_k}}\} \) of the gradient flow extensions from the level \( 2c_{n_k} \) of the minimizers \( \{q_{c_n}\} \) (see Definition 3.3) such that for \( q_k := q_{c_{n_k},2c_{n_k}}(\cdot - t_k) \) one has

\[ q_k - U_0 \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R};\mathbb{R}^N) \text{ as } k \to \infty, \]

where \( U_0 \) is a minimizer of (HCP), in particular, a heteroclinic connection between the wells of \( W \).

**Remark 3.8.** The proof of Theorem 3.1 given below will actually reveal additional types of convergence of the sequence \( \{q_k\} \) to a heteroclinic connection:

\[ q_k \xrightarrow[k \to \infty]{} U_0 \text{ in } C_0^{0,\alpha}(\mathbb{R};\mathbb{R}^N) \text{ for any } \alpha \in (0,1/2), \text{ and } \]

\[ q_k \xrightarrow[k \to \infty]{} U_0 \text{ a.e. on } \mathbb{R}. \]

**Proof of Theorem 3.1.** The content of (i) is precisely the one of Theorem 3.2, proved in §3.5. On the other hand, before arguing (ii) we need to make the following observation.

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Lemma 3.10. Let us write $T_t: H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \to H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$ for the operator of translation in time by $t$, $T_tq(s) = q(s-t)$, and $G_\beta : \cup_{c \in (0, \beta)} M_c \to H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N)$ for the operator of extension by gradient flow from level $\beta$. Then, these operators commute:

$$(G_\beta \circ T_t)(q_c) = (T_t \circ G_\beta)(q_c) \quad \text{for } q_c \in M_c,$$

provided $t_c$ satisfies (3.19) and $0 < c < \beta < c_*$ holds.

Proof of Lemma 3.10. The Definition 3.1 of the control times $\tau_{c,\beta}^\pm$ shows that $(\tau_{c,\beta}^-, \tau_{c,\beta}^+)$ can be characterized as the maximal open interval around 0 where

$$W(q_c(t)) > \beta \quad \text{for all } t \in (\tau_{c,\beta}^-, \tau_{c,\beta}^+).$$

In other words, $(\tau_{c,\beta}^- + t_c, \tau_{c,\beta}^+ + t_c)$ is the maximal open interval around 0 where

$$W(T_tq_c(t)) > \beta \quad \text{for all } t \in (\tau_{c,\beta}^- + t_c, \tau_{c,\beta}^+ + t_c). \quad (3.68)$$

If we denote $(\tilde{\tau}_{c,\beta}^-, \tilde{\tau}_{c,\beta}^+)$ the maximal interval of $t \mapsto T_tq_c(t)$ where (3.68) holds, then the uniqueness of the endpoints of such intervals (see Proposition 3.1) shows $\tilde{\tau}_{c,\beta}^\pm = \tau_{c,\beta}^\pm + t_c$. Therefore, using the Definition 3.3 of gradient flow extension we readily check,

$$G_\beta[T_tq_c](t) := \begin{cases} 
\Phi_1(t - \tilde{\tau}_{c,\beta}^-, b^-), & t \in (-\infty, \tilde{\tau}_{c,\beta}^-), \\
T_tq_c(t), & t \in [\tilde{\tau}_{c,\beta}^-, \tilde{\tau}_{c,\beta}^+], \\
\Phi_2(t - \tilde{\tau}_{c,\beta}^+, b^+), & t \in [\tilde{\tau}_{c,\beta}^+, +\infty).
\end{cases}$$

$$= \begin{cases} 
\Phi_1(t - t_c - \tau_{c,\beta}^-, b^-), & t - t_c \in (-\infty, \tau_{c,\beta}^-), \\
q_c(t - t_c), & t - t_c \in [\tau_{c,\beta}^-, \tau_{c,\beta}^+], \\
\Phi_2(t - t_c - \tau_{c,\beta}^+, b^+), & t - t_c \in [\tau_{c,\beta}^+, +\infty).
\end{cases}$$

$$= [G_\beta q_c](t - t_c) = T_t[G_\beta q_c](t).$$

Let us write $q_n := q_{c_n, 2c_n}(\cdot - t_{c_n})$ as defined in the statement of Theorem 3.1, and also we recall the notation $q_{c_n, 2c_n}$ for the gradient flow extension from level $2c_n$ of the time translates.
\[ q_{cn}(t) = q_{cn}(t - t_{cn}), \] as defined in (3.19). In light of Lemma 3.10 we see that
\[ q_{n}(t) := T_{t_{cn}}[G_{2cn}q_{cn}](t) = G_{2cn}[T_{t_{cn}}q_{cn}](t) =: q_{cn, 2cn}(t) \quad \text{for all } n \text{ large, and } t \in \mathbb{R}. \] (3.69)

Having the identity (3.69) in mind, we now conclude this proof by studying the sequence \( \{q_{cn, 2cn}\} \) instead of writing \( q_{n} \), so we are consistent with the notation of the results in the previous sections.

Now we proceed to prove (ii). Owing to (3.58)-(3.59) in Proposition 3.4 we see that for any sequence \( c_{n} \rightarrow 0^{+} \) there exists \( n_{0} > 1 \) sufficiently large, so that
\[
\sup_{n \geq n_{0}} \|q_{cn, 2cn} - U_{aff}\|_{H^{1}(\mathbb{R}; \mathbb{R}^{N})}^{2} \leq |p_{+} - p_{-}|^{2} + \frac{M_{0}}{\sigma_{0}} + \frac{4M_{0}}{\beta_{0}} \left( \beta_{0} + 3 \max_{\{W \leq c_{*}\}} |u|^{2} + 2 \max \left\{ \frac{c_{*}}{\sigma_{0}}, \frac{4M_{0}^{2}}{c_{*}} \right\} \right),
\]
where \( n_{0} \) is chosen so \( c_{n} \in (0, \min\{M_{0}/2, c_{*}\}) \); that is, the sequence \( \{q_{cn, 2cn} - U_{aff}\} \) is bounded in \( H^{1}(\mathbb{R}; \mathbb{R}^{N}) \). Owing to the reflexivity of \( H^{1}(\mathbb{R}; \mathbb{R}^{N}) \), there must exist \( \tilde{U}_{0} \in H^{1}(\mathbb{R}; \mathbb{R}^{N}) \) so that, up to subsequences,
\[
q_{cn, 2cn} - U_{aff} \rightharpoonup \tilde{U}_{0} \text{ weakly in } H^{1}(\mathbb{R}; \mathbb{R}^{N}) \text{ as } n \rightarrow \infty. \] (3.70)

Put equivalently, if we let \( U_{0} := U_{aff} + \tilde{U}_{0} \in H_{aff}^{1}(\mathbb{R}; \mathbb{R}^{N}) \) and if we continue to index the desired subsequence by \( c_{n} \), then
\[
q_{cn, 2cn} - U_{0} \rightharpoonup 0 \text{ weakly in } H^{1}(\mathbb{R}; \mathbb{R}^{N}) \text{ as } n \rightarrow \infty. \] (3.71)

On the other hand, (3.57) in Proposition 3.4 shows that
\[
\sup_{n \geq n_{1}} \|q_{cn, 2cn} - U_{aff}\|_{C^{0,1/2}(\mathbb{R}; \mathbb{R}^{N})} \leq \sqrt{2M_{0}} + \max_{\{u : W(u) \leq c_{*}\}} |u| + \max \left\{ \sqrt{\frac{c_{*}}{\sigma_{0}}}, M_{0}, \sqrt{\frac{2}{c_{*}}} \right\} + \|U_{aff}\|_{C^{0,1/2}(\mathbb{R}; \mathbb{R}^{N})},
\]
for \( n_{1} \) chosen sufficiently large. Hence, the Arzela-Ascoli theorem together with a diagonalization argument shows that the sequence \( \{q_{cn, 2cn} - U_{aff}\} \) is precompact in the topology of the uniform convergence on compact subsets of \( \mathbb{R} \). This uniform convergence implies weak convergence.
of \( \{ q_{c_n,2c_n} - U_{\text{aff}} \} \) in \( L^2 \) on compact sets, whence the uniqueness of the weak \( L^2 \)-limit together with (3.70) shows

\[
q_{c_{n_k},2c_{n_k}} - U_{\text{aff}} \rightarrow \tilde{U}_0 \quad \text{in} \quad C^0_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \quad \text{as} \quad k \to \infty,
\]

(3.73)

for a subsequence \( \{ c_{n_k} \} \) of \( \{ c_n \} \). In other words, by recalling the definition of \( U_0 \) we conclude there exists a subsequence of \( \{ q_{c_n,2c_n} \} \) converging to a limit \( U_0 \in H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \) in various senses

\[
\begin{align*}
q_{c_{n_k},2c_{n_k}} & \xrightarrow{k \to \infty} U_0 \quad \text{in} \quad C^0_{\text{loc}}(\mathbb{R}; \mathbb{R}^N), \\
q_{c_{n_k},2c_{n_k}} & \xrightarrow{k \to \infty} U_0 \quad \text{a.e. on} \quad \mathbb{R}.
\end{align*}
\]

(3.74)

Thus we are only left to verify that \( U_0 \) is a heteroclinic connection between \( p_\pm \), minimizer of (HCP). Using (3.71) and the weak lower semi-continuity of the \( L^2 \)-norm, plus (3.74) together with the Fatou’s lemma (since \( W \geq 0 \)), we obtain

\[
m_0 \leq H^0(U_0) = \int_{-\infty}^{+\infty} |U_0'|^2 + W(U_0) \leq \liminf_{k \to \infty} \int_{-\infty}^{+\infty} |q_{c_{n_k},2c_{n_k}}'|^2 + W(q_{c_{n_k},2c_{n_k}}) \leq \liminf_{k \to \infty} H^0(q_{c_{n_k},2c_{n_k}}) = m_0,
\]

in view of the fact that \( \{ q_{c_n,2c_n} \} \) is a minimizing sequence for (HCP) (see Corollary 3.6). Whence, \( U_0 \) is a minimizer of \( H^0 \) in \( H^1_{\text{aff}}(\mathbb{R}; \mathbb{R}^N) \) and it a solves classically the connection problem

\[
U_0'' = \nabla W(U_0), \quad U_0(\pm \infty) = p_\pm.
\]

(3.75)

This finishes the proof of item (ii). Finally, let us observe that the convergence of \( \{ q_{c_{n_k},2c_{n_k}} \} \) in (3.74) can be improved by using the \( C^{0,1/2} \)-bound (3.72) combined with the compact embedding \( C^{0,1/2}([-k,k]) \subset C^{0,\alpha}([-k,k]) \) for any \( k \in \mathbb{N} \) and \( 0 < \alpha < 1/2 \) (see e.g. [1, THM 1.34]). Indeed, a diagonalization argument and the uniqueness of limits, just like shown before, then yields

\[
q_{c_{n_k},2c_{n_k}} \xrightarrow{k \to \infty} U_0 \quad \text{in} \quad C^0_{\text{loc}}(\mathbb{R}; \mathbb{R}^N),
\]

for any \( \alpha \in (0,1/2) \), and the proof Theorem 3.1 is now complete. \( \square \)

**Remark 3.9.** A linearization analysis of the first order semilinear system associated to (3.75),

\[
\frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} Y \\ W(p_+ + X) \end{bmatrix}, \quad \lim_{t \to +\infty} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X(t) = U_0(t) - p_+
\]

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(analogously $X(t) := U_0(t) - p_-$ for $t \to -\infty$) can be used to show that the rate of convergence of the classical solution $U_0$ to the wells $p_\pm$ can be improved to be exponential as $t \to \pm \infty$, for potentials satisfying (W1) through (W4).
Part II

Study of weighted least gradient problems

Chapter 4

Continuity of minimizers to weighted least gradient problems

4.1 Introduction and statement of the main Theorem 4.1

In this chapter we revisit the question of existence and regularity of solutions in higher dimensions to weighted least gradient problems subject to a Dirichlet boundary condition

\[
\inf \left\{ \int_{\Omega} a(x)|Du| : u \in BV(\Omega), u|_{\partial\Omega} = g \right\},
\]

(4.1)

where \( g \in C(\partial\Omega) \), and the weight function \( a \) is bounded away from zero with \( a \in C^3(\overline{\Omega}) \).

Let us now formulate the problem more precisely. Given \( N \geq 2 \) arbitrary, a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^N \), and \( a \in C^3(\overline{\Omega}) \) a weight function satisfying the following non-degeneracy condition

\[
\min_{\overline{\Omega}} a \geq \alpha > 0,
\]

(4.2)

for some \( \alpha \in (0, \infty) \), we deal with the study of minimizers of the weighted \( a \)-variation functional over the set of \( BV(\Omega) \) functions that coincide on the boundary with some data \( g : \partial\Omega \to \mathbb{R} \) in the sense of \( BV \)-traces. That is,

\[
\inf_{u \in BV_g(\Omega)} \int_{\Omega} a(x)|Du|,
\]

(\( a \)LGP)
where the admissible class is defined via

\[ BV_g(\Omega) := \{ u \in BV(\Omega) : \lim_{r \to 0} \text{ess sup}_{y \in \Omega \cap |x-y| < r} |u(y) - g(x)| = 0 \text{ for all } x \in \partial\Omega \}. \quad (4.3) \]

Here \( BV(\Omega) \) denotes the class of functions of bounded variation in \( \Omega \) (see [30]).

Let us recall the notion of \( a \)-variation of \( u \in BV(\Omega) \) induced by the continuous function \( a : \Omega \to (0, \infty) \), uniformly bounded away from zero. As introduced by Amar and Belletini in [10], the \( a \)-variation of \( u \in BV(\Omega) \) in \( \Omega \) is given by

\[ \hat{U}_a(x) |Du| := \sup \left\{ \int_U u \text{div} Y \, dx : Y \in C^\infty_c(U; \mathbb{R}^N), |Y(x)| \leq a(x) \text{ for all } x \in \Omega \right\}. \quad (4.4) \]

This corresponds to the definition of \( \phi \)-variation of \( u \) in [10] for the choice of \( \phi(x, \xi) = a(x)|\xi| \), which is described in terms of the dual norm \( \phi^0(x, \xi) := \sup\{ \xi \cdot p : \phi(x, p) \leq 1 \} \). In (4.4) we have used the fact that \( \phi^0(x, \xi) = |\xi|/a(x) \) for such choice of an inhomogeneous, isotropic norm \( \phi \). This notion gives rise to a Radon measure on \( \mathbb{R}^N \) induced by \( u \) that acts on Borel sets via \( B \mapsto \int_B a(x)|Du| \), called the \( a \)-variation measure of \( u \). By analogy, we define the \( a \)-perimeter of a Caccioppoli set \( E \subset \mathbb{R}^N \) as the Radon measure that on any Borel set \( B \) assigns the value

\[ \mathcal{P}_a(E, B) := \int_B a(x)|D\chi_E|, \]

where \( \chi_E \) is the characteristic function of \( E \).

The main concern of Chapter 4 will be to establish the existence of a continuous minimizer of \( (a\text{LGP}) \) even in the possible presence of singularities for the level sets of the solution, when continuous boundary data \( g \in C(\partial\Omega) \) is considered and for a class of domains satisfying suitable geometric conditions. Specifically, we will require that \( \Omega \) is a bounded Lipschitz domain with connected boundary, which in addition fulfills the following

**Condition 2 (Barrier condition).** For every \( x_0 \in \partial\Omega \) there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) if \( V_\varepsilon \subset \Omega \) is a minimizer of

\[ \inf\{ \mathcal{P}_a(W, \mathbb{R}^N) : W \subset \Omega, \Omega \setminus W \subset B_\varepsilon(x_0) \}, \quad (4.5) \]
then
\[ \partial V^* \cap \partial \Omega \cap B_\varepsilon(x_0) = \emptyset. \]

The boundaries of such domains \( \Omega \) are not locally \( \alpha \)-area minimizing with respect to interior variations (cf. [39]). Nonetheless, it has been recently pointed out by Spradlin and Tamasan in [77] that, even for domains \( \Omega \) satisfying the barrier condition (4.5), the existence of minimizers to
\[
\inf \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), u|_{\partial \Omega} = g \right\}
\]
may fail for some choices of discontinuous boundary data \( g \).

An existence and continuity result of minimizers was already established by Jerrard, Moradifam and Nachman in [39] for a more general version of the least gradient problem
\[
\inf_{u \in BV(\Omega)} \int_{\Omega} \varphi(x, Du), \quad (\varphi\text{LGP})
\]
for the admissible class given in (4.3), \( g \in C(\partial \Omega) \), and a function \( \varphi(x, \xi) \) that, among other properties is convex, continuous, and 1-homogeneous with respect to the \( \xi \)-variable. They prove existence and comparison results (uniqueness) for \((\varphi\text{LGP})\) valid in all dimensions \( N \geq 2 \) for domains \( \Omega \) satisfying a barrier condition suited to a general class of inhomogeneous anisotropic \( \varphi \)-perimeter functionals. In contrast, their regularity theorem established for \((\varphi\text{LGP})\), under sharp conditions, is valid in low dimensions \( N = 2, 3 \) only. In a related work, Moradifam has argued that the structure of the level sets of minimizers to \((\varphi\text{LGP})\) are determined by a divergence free vector-field (see [54] for a precise statement).

Despite the dimensionality restriction of the regularity result in [39], it is nonetheless the case that when \((\alpha\text{LGP})\) is considered, i.e. \( \varphi(x, \xi) = \alpha(x)|\xi| \), their result applies up to dimension \( N \leq 7 \) by virtue of the regularity theory of minimal hypersurfaces, with respect to an area functional induced by a Riemannian metric (see Remark 4.8 in [39] and references therein). In light of this, a major thrust of the present paper is to establish such a continuity result for a minimizer of \((\alpha\text{LGP})\) in higher dimensions \( N \geq 8 \).
The approach we will adopt in this article consists of applying the Sternberg-Williams-Ziemer program in [80] to construct continuous minimizers of the weighted least gradient problem subject to a Dirichlet boundary condition. In fact, a secondary reason for this investigation has been to determine whether this technique carries over to the setting of weighted least gradient problems. Their method is based on the co-area formula and on an auxiliary geometric variational problem to identify the level sets of such minimizers. Indeed, in [18] it was shown that the superlevel sets of a continuous function of least gradient are area-minimizing, that is, the characteristic functions of those sets are functions of least gradient. Conversely, the authors in [80] proved the existence and continuity of a function of least gradient for every dimension $N \geq 2$, by explicitly constructing each of its superlevel sets in such a way that they are area-minimizing and reflect the boundary condition, as long as two geometric conditions of $\partial \Omega$ are satisfied, referred as a weak non-negative mean curvature condition and the assumption that $\partial \Omega$ is not locally area-minimizing with respect to interior set variations. Their proof relies, among other things, on a strict maximum principle for area-minimizing sets established by Simon in [72].

We now state the main result of Chapter 4.

**Theorem 4.1.** For any $N \geq 2$, let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with Lipschitz boundary satisfying the barrier condition (4.5), and let $a \in C^3(\bar{\Omega})$ be a non-degenerate function in the sense of (4.2). Then for any boundary data $g \in C(\partial \Omega)$, there exists a minimizer $u$ to $(aLGP)$, which is moreover a continuous function $u \in C(\bar{\Omega})$. Furthermore, the superlevel sets of this function minimize the weighted perimeter measure $\mathcal{P}_a(\cdot, \Omega)$ with respect to competitors meeting the boundary conditions imposed by $g$ on $\partial \Omega$.

### 4.2 Notation and preliminaries

Let us write $B_r(x)$ for the open Euclidean ball centered at $x \in \mathbb{R}^N$ of radius $r > 0$, and we abbreviate $B_R := B_R(0)$, unless otherwise specified. The notation $B'_r(x')$ will be reserved for balls
in $\mathbb{R}^{N-1}$ centered at $x' \in \mathbb{R}^{N-1}$, where we will consistently write $x = (x', x'') \in \mathbb{R}^{N-1} \times \mathbb{R}$ for points $x \in \mathbb{R}^N$. With a slight abuse of notation we let $| \cdot |$ refer to the Euclidean distance between points in $\mathbb{R}^N$ as well as to the Lebesgue measure in $\mathbb{R}^N$. In addition, $\mathcal{H}^\alpha$ corresponds to the $\alpha$-dimensional Hausdorff measure in $(\mathbb{R}^N, | \cdot |)$. Throughout, we will primarily employ $\mathcal{H}^{N-1}$. On the other hand, given a set $E \subset \mathbb{R}^N$, $E^i$ denotes the topological interior of $E$, $\bar{E}$ denotes the topological closure of $E$, and $\partial E$ denotes its topological boundary. Also, the notation $E \subset \subset F$ refers to the containment $E \subset F^i$. We recall the measure-theoretic boundary of $E$,

$$\partial_M E := \{ x \in \mathbb{R}^N : \tilde{\Theta}(E, x) > 0 \} \cap \{ x \in \mathbb{R}^N : \Theta(E, x) < 1 \},$$

where

$$\tilde{\Theta}(E, x) := \limsup_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|},$$

$$\Theta(E, x) := \liminf_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|},$$

are the upper and lower densities of $E$ at $x$, respectively. Moreover, the reduced boundary of $E$ is the set $\partial^* E = \{ x : \nu_E(x) \text{ exists} \}$, where $\nu_E(x)$ is the so-called measure theoretic normal of the set $E$, defined as the unique vector $\nu \in \mathbb{R}^N$ satisfying

$$\tilde{\Theta}(\{ y : (y - x) \cdot \nu > 0, \ y \in E \}, x) = 0 \quad \text{and} \quad \tilde{\Theta}(\{ y : (y - x) \cdot \nu < 0, \ y \notin E \}, x) = 0.$$

It is well-known that

$$\partial^* E \subset \partial_M E \subset \partial E. \quad (4.6)$$

Moreover, $E$ is of finite perimeter if and only if $\mathcal{H}^{N-1}(\partial_M E) < \infty$; and in this case

$$\mathcal{P}(E, \Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial_M E) = \mathcal{H}^{N-1}(\Omega \cap \partial^* E),$$

cf. [27]. Throughout, we employ the measure theoretic closure to represent the equivalence class of sets of finite perimeter, which differ only up to sets of $\mathcal{H}^N$-measure zero. With this convention, we let

$$x \in E \quad \text{if and only if} \quad \tilde{\Theta}(E, x) > 0. \quad (4.7)$$
It can be shown using this convention (4.7) that $\partial^* E = \partial E$, cf. [30, Thm 4.4].

Given a Radon measure $\mu$ in a locally compact topological space $M$ and $f \in L^1_{loc}(M, \mu)$, we adopt a notation already introduced in [25,73], where $\mu \ll f$ denotes the Radon measure acting on Borel sets of $M$ via

$$
\mu \ll f(A) := \int_A f(x) \, d\mu(x).
$$

Let us now review some basic properties of functions of bounded $a$-variation, and of sets of finite $a$-perimeter, within the class of continuous weights $a \in C(\bar{\Omega})$ satisfying $a > 0$. For a proof of the facts below, we refer the reader to [10,39].

The theory of inhomogeneous and anisotropic variations rests upon the following integral representation formula.

**Proposition 4.1** ([10, Prop. 7.1]). For any $u \in BV_{loc}(\mathbb{R}^N)$ and a bounded Borel set $B$,

$$
\int_B a(x)|D(u)| = \int_B a(x)|\sigma^u(x)| \, d|Du|(x) = \int_B a(x) \, d|Du|(x),
$$

where $\sigma^u := dDu/d|Du| \in \mathbb{R}^N$ denotes the Radon-Nikodým density. Here we have used that $|\sigma^u| = 1$ for $|Du|$-a.e $x \in B$ (see [25, §5]).

An immediate corollary of this proposition, using the characterization of the perimeter measure of Caccioppoli sets [30, §4], is the fact that for any Borel set $B$

$$
\mathcal{P}_a(E, B) = \mathcal{H}^{N-1}(\partial^* E \cap B),
$$

where $\mathcal{H}^{N-1}(\partial^* E \cap B) := \int_{\partial^* E \cap B} a(x) \, d\mathcal{H}^{N-1}(x)$.

The integral representation formula (4.8) of the $a$-variation, together with the Fleming-Rishel co-area formula for $BV$ functions imply a weighted version of the co-area formula:

**Proposition 4.2** ([10, Rem. 4.4]). If $u \in BV_{loc}(\mathbb{R}^N)$ and $B \subset \mathbb{R}^N$ is Borel, then

$$
\int_B a(x)|D(u)| = \int_{-\infty}^{+\infty} \mathcal{P}_a(\{u \geq t\}, B) \, dt.
$$
Furthermore, just like for the standard perimeter measure, the following inequality holds true as well for the $a$-perimeter functional

**Proposition 4.3** ([39, Lem. 2.2]). For $B \subset \mathbb{R}^N$ Borel and $E_1, E_2 \subset \mathbb{R}^N$ sets of locally finite $a$-perimeter,

$$\mathcal{P}_a(E_1 \cup E_2, B) + \mathcal{P}_a(E_1 \cap E_2, B) \leq \mathcal{P}_a(E_1, B) + \mathcal{P}_a(E_2, B).$$

The following lower semi-continuity property of $a$-perimeter will be essential in our development, whose proof follows from a standard argument in the $BV$ theory.

**Proposition 4.4.** Let $U \subset \mathbb{R}^N$ be an open set, $a \in C(\bar{U})$ a weight function with $a > 0$, and \{${u}_j$\} $\subset BV(U)$ a sequence of functions that converge in $L^1_{loc}(U)$ to a function $u$. Then $u$ has finite $a$-variation in $U$, and moreover

$$\int_U a(x)|Du| \leq \liminf_{j \to \infty} \int_U a(x)|Du_j|.$$ 

Of particular importance to us are sets of finite $a$-perimeter whose boundaries minimize the weighted area $\mathcal{H}^{N-1}\lfloor a$, that we will refer from now on as the $a$-area measure.

**Definition 4.1.** If $E$ is a set of locally finite $a$-perimeter and $U$ is a bounded, open set, we say that $\partial E$ is $a$-area minimizing in $U$ if

$$\mathcal{P}_a(E, U) = \inf \{\mathcal{P}_a(F, U) : E \Delta F \subset U\}, \quad (4.10)$$

Also, we say $\partial E$ is locally $a$-area minimizing if (4.10) holds true for every choice of open, bounded subset $U$ of $\mathbb{R}^N$.

The regularity of $\partial E$ will play a crucial role in our development just like in the standard theory. Tangent cones are the building blocks of the regularity theory in the inhomogeneous setting treated here as well. More precisely, if $\partial E$ is $a$-area minimizing in a bounded open set $U \subset \mathbb{R}^N$, then for each $x \in \partial E$ and each sequence $\lambda_j \to 0^+$ the exist a subsequence $\{\lambda_{j'}\} \subset \{\lambda_j\}$ and a Borel set $F$
of locally finite $\alpha$-perimeter, so that $\partial F$ is $\alpha$-area minimizing in $U$ and if we denote the translation plus homothety $E_j := \{ y \in \mathbb{R}^N : x + \lambda_j(y - x) \in E \}$, it follows $\chi_{E_j} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^N)$. Here $C = \partial F$ is called tangent cone to $E$ at $x$ (not unique a priori). Although it is not immediate, $C$ is a union of half-lines issuing from $x$ and thus is a cone. It follows from [71, Thm. I.1.2] that if $C$ is contained in any hyperplane of $\mathbb{R}^N$ supported at $x$, then the tangent cone of $\partial E$ at $x$ is unique, and $\partial E$ is regular at $x$. Namely, there exists $r > 0$ so that $B_r(x) \cap \partial E$ is a $C^2$-hypersurface. See Remark 4.1 in §4.3 for additional comments.

Furthermore, let us denote $\text{reg}(\partial E) := \{ x \in \partial E : \partial E \text{ is regular at } x \}$ and $\text{sing}(\partial E) := \partial E \setminus \text{reg}(\partial E)$. In [71, Thm I.3.1, Cor. I.3.2] it has been proved, in particular, that for a set $E$ with $\partial E$ minimizing the $\alpha$-area in $U$,

\[
\begin{cases}
\mathcal{H}^{N-3}(\text{sing}(\partial E) \cap U) < \infty, & \text{if } N \geq 4 \\
\text{sing}(\partial E) \cap U = \emptyset, & \text{if } N \leq 3 .
\end{cases}
\] (4.11)

It follows that $\text{reg}(\partial E)$ is dense in $\partial E$.

4.3 A strict maximum principle for $\alpha$-perimeter minimizing sets Theorem 4.2

This aim of the current section is to present a general maximum principle for $\alpha$-perimeter minimizing sets, and to provide a full proof of it. It states that when two sets are nested and touch at a point, and have $\alpha$-minimizing perimeter, then they should locally coincide. More precisely, this result states that

**Theorem 4.2 (Maximum principle for $\alpha$-area minimizing sets).** Let $E_1 \subset E_2$ be subsets in $\mathbb{R}^N$ where both $\partial E_1$, $\partial E_2$ minimize the $\alpha$-area in an open set $U \subset \mathbb{R}^N$. If $x \in \partial E_1 \cap \partial E_2 \cap U$, then $\partial E_1$ and $\partial E_2$ agree in some neighborhood of $x$.

This is an adaptation of the analogous Theorem 1 in [80], to this particular setting of an inhomogeneous isotropic Riemannian metric. The contribution of such result, when compared to the classical strong maximum principle for minimal surfaces, is that it includes the case of
hypersurfaces that may contain singularities, around which the hypersurface cannot be written as the graph of a function over the tangent plane based at the singularity point. In particular, the contact point $x$ mentioned above could potentially be a singular point for either of $E_1$ or $E_2$. This problematic situation in the context of minimal surfaces has been resolved by Leon Simon in a celebrated result, known as the strict maximum principle for mass minimizing currents, of fundamental importance in geometric measure theory. It deals with a more general situation than the one mentioned in the above theorem, where the main object of study are currents.

**Theorem 4.3** ([72, Thm. 1]). Let $U$ be an open set of a smooth $N$-dimensional oriented Riemannian manifold $N$. Suppose $T_1$ and $T_2$ are integer multiplicity currents with $\partial T_1 = 0 = \partial T_2$ in $U$, with $T_1, T_2$ mass-minimizing in $U$, and $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 \cap U = \emptyset$. Then,

$$\operatorname{supp} T_1 \cap \operatorname{supp} T_2 \cap U = \emptyset.$$

The main content of this theorem lies in the fact that $\operatorname{sing} T_1 \cap \operatorname{sing} T_2 \cap U = \emptyset$. Indeed, previous work in [48], and also in [73, §37.10] establishes $\operatorname{sing} T_1 \cap \operatorname{reg} T_2 \cap U = \emptyset$. Subsequently, this latter fact was proved in [75], even without the minimizing hypothesis. An immediate consequence of Theorem 4.3 is the following corollary for oriented boundaries of least area:

**Corollary 4.1** ([72, Cor. 1]). Adopting the notation of Theorem 4.3, let $T_1 = (\partial [E_1]) \mathcal{L} U$, $T_2 = (\partial [E_2]) \mathcal{L} U$ be mass-minimizing currents in $U$, with $E_1 \cap U \subset E_2 \cap U$ and with $\operatorname{supp} T_1 \cap U$ and $\operatorname{supp} T_2 \cap U$ connected. Then either

$$T_1 = T_2 \quad \text{or} \quad \operatorname{supp} T_1 \cap \operatorname{supp} T_2 \cap U = \emptyset.$$

Ultimately, this tool allows us to push our continuity result to all dimensions $N \geq 2$, and in particular to the case $N \geq 8$, where $\alpha$-perimeter minimizing sets cease to be smooth in general, and the strong maximum principle from PDE is no longer available (even in the weak form). For details, the reader can compare to the continuity result in [39].
The proof of Theorem 4.2 is rather technical, and it uses different ingredients in geometric analysis, the main one being Corollary 4.1. The purpose of this section is to present a collection of results along with their proofs, that will give a proof of Theorem 4.2.

4.3.1 A strict maximum principle for regular \( \alpha \)-minimal hypersurfaces

The following is a weak version of the strict maximum principle for hypersurfaces that minimize the \( \alpha \)-area functional, in the case where the intersection point is regular for both hypersurfaces. In this situation we have PDE techniques available, and the next result corresponds to a weak version of the Hopf maximum principle for quasilinear equations in divergence form. Before introducing this result, we need to discuss some preliminary concepts.

Given a Caccioppoli set \( E \) and \( x_0 \in \partial^* E \) it is known that \( \partial E \) can be locally represented as the graph of a non-negative \( C^1 \)-function \( u \) over a tangent plane at \( x_0 \), cf. [30]. This means that up to isometries, if we write \( (x', x'') \in \mathbb{R}^{N-1} \times \mathbb{R} \) for the coordinates of \( x \in \mathbb{R}^N \), and if \( B'_r(x') \) denotes a ball in \( \mathbb{R}^{N-1} \), then is a choice of \( r = r(x_0) > 0 \) small and \( u \in C^1(B'_r(x_0)) \) non-negative in such a way that

\[
\{(x', s) : x' \in B'_r(x_0), \ 0 \leq s \leq u(x')\} \subset E,
\]

and

\[
\partial E \cap B_r(x_0) = \{(x', u(x')) : x' \in B'_r(x_0)\}.
\]

In particular, the characterization (4.9) of the \( \alpha \)-perimeter measure together with the area formula (cf. [25, Thm. 4.1]) allows us to compute in local coordinates

\[
\mathcal{P}_\alpha(E, B_r(x_0)) = \int_{B'_r(x_0)} \alpha(x', u(x')) \sqrt{1 + |\nabla u(x')|^2} \, d\mathcal{H}^{N-1}(x') =: I_\alpha(u).
\] (4.12)

Here and henceforth, we denote \( \nabla := \nabla_{x'} \) for the gradient in \( \mathbb{R}^{N-1} \), whenever appropriate. Com-
puting the first variation of \( I_\alpha(u) \) in the direction of \( \varphi \), \( \delta I_\alpha(u)[\varphi] \), we obtain

\[
M_\alpha u(\varphi) := \int_{B'_r(x'_0)} \left\{ \alpha(x', u) \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla \varphi + \partial_s \alpha(x', u) \sqrt{1 + |\nabla u|^2} \varphi \right\} dx' = 0. \tag{4.13}
\]

This operator will be called the \( \alpha \)-minimal surface operator of \( u \), acting on any test function \( \varphi \) in \( B'_r(x'_0) \). The last observation motivates the following notions

**Definition 4.2.** Let \( W \subset \mathbb{R}^{N-1} \) be an open set. A function \( u \in C^1(W) \) is called a classical solution to the inhomogeneous \( \alpha \)-minimal surface equation (or \( \alpha \)-MSE) in \( W \), if

\[
-\text{div}_{x'} \left( \alpha(x', u) \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + \partial_s \alpha(x', u) \sqrt{1 + |\nabla u|^2} = 0 \quad \text{for} \quad x' \in W.
\]

Also, \( u \in C^1(W) \) is called a weak subsolution (weak supersolution) of the inhomogeneous \( \alpha \)-MSE in \( W \) if

\[
M_\alpha u(\varphi) := \int_W \left\{ \alpha(x', u) \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla \varphi + \partial_s \alpha(x', u) \sqrt{1 + |\nabla u|^2} \varphi \right\} dx' \leq 0 \quad (\geq 0),
\]

whenever \( \varphi \in C^1_c(W) \) with \( \varphi \geq 0 \).

The next result establishes the maximum principle we require, in the favorable situation where the contact point is regular for the two touching hypersurfaces.

**Proposition 4.5 (Weak form of strict maximum principle).** Let \( W \) be an open set in \( \mathbb{R}^{N-1} \) and let \( u_1, u_2 \in C^1(W) \) be a weak supersolution and a weak subsolution of the inhomogeneous \( \alpha \)-MSE in \( W \), respectively. Suppose that \( u_2 \geq u_1 \) in \( W \), while \( u_2(x'_0) = u_1(x'_0) \) for some \( x'_0 \in W \). Then \( u_2 = u_1 \) in some open neighborhood of \( x'_0 \) in \( W \).

**Proof Proposition 4.5.** Let us write \( w(x') := u_2(x') - u_1(x') \) and \( u'(x') := u_1(x') + tw(x') \) for \( t \in [0, 1] \), and \( x' \in W \). It will be convenient for computations to rewrite the \( \alpha \)-minimal surface operator acting on \( \varphi \in C^1_c(W) \) with \( \varphi \geq 0 \), as

\[
M_\alpha u(\varphi) = \int_W (f(x', u, \nabla u) \cdot \nabla \varphi + f(x', u, \nabla u)\varphi)dx',
\]
where for \( x', p' \in \mathbb{R}^{N-1} \) and \( s \in \mathbb{R} \) we let

\[
  f(x', s, p') := a(x', s) \frac{p'}{\sqrt{1 + |p'|^2}}, \quad f(x', s) := \partial_s a(x', s) \sqrt{1 + |p'|^2}.
\]

Using the linearity of \( \varphi \mapsto \mathcal{M}_a u(\varphi) \), we compute the difference \( \mathcal{M}_a u_1(\varphi) - \mathcal{M}_a u_0(\varphi) \) by means of the chain rule

\[
  f(x', u_2, \nabla u_2) - f(x', u_1, \nabla u_1) = \int_0^1 \frac{d}{dt} f(x', u^t, \nabla u^t) dt = \{a^{ij}(x') \partial_j w + b^i(x') w\}_{i=1, \ldots, N-1},
\]

with coefficients

\[
  a^{ij}(x') := \int_0^1 \left[ D_p f(x', u^t, \nabla u^t) \right]_{ij} dt = \int_0^1 a(x', u^t) \frac{(1 + |\nabla u^t|^2) \delta_{ij} - \partial_i u^t \partial_j u^t}{(1 + |\nabla u^t|^2)^{3/2}} dt,
\]

\[
  b^i(x') := \int_0^1 \left[ \partial_s f(x', u^t, \nabla u^t) \right]_i dt = \int_0^1 \partial_s a(x', u^t) \frac{\partial_i u^t}{\sqrt{1 + |\nabla u^t|^2}} dt.
\]

Similarly one has

\[
  f(x', u_2, \nabla u_2) - f(x', u_1, \nabla u_1) = \int_0^1 \frac{d}{dt} f(x', u^t, \nabla u^t) dt = -(c^j(x') \partial_j w + d(x') w),
\]

where

\[
  c^j(x') := -\int_0^1 \left[ D_p f(x', u^t, \nabla u^t) \right]_j dt = \int_0^1 -\partial_s a(x', u^t) \frac{\partial_j u^t}{\sqrt{1 + |\nabla u^t|^2}} dt,
\]

\[
  d(x') := -\int_0^1 \partial_s f(x', u^t, \nabla u^t) dt = \int_0^1 -\partial_{ss} a(x', u^t) \sqrt{1 + |\nabla u^t|^2} dt.
\]

Now, since \( u_1, u_2 \) are weak supersolution and subsolution of \( a\)-MSE respectively, we get

\[
  0 \leq \mathcal{M}_a u_2(\varphi) - \mathcal{M}_a u_1(\varphi)
  = \int_W \{(f(x', u_2, \nabla u_2) - f(x', u_1, \nabla u_1)) \cdot \nabla \varphi + (f(x', u_2, \nabla u_2) - f(x', u_1, \nabla u_1)) \varphi\} dx'
  = \int_W \{ (a^{ij}(x') \partial_j w + b^i(x') w) \partial_\iota \varphi - (c^j(x') \partial_j w + d(x') w) \varphi \} dx'
  =: \mathcal{L}(w, \varphi).
\]

Thus, \( w \) is a weak supersolution of \( Lw = 0 \), where \( L \) corresponds to the linear operator in divergence form

\[
  Lw := \partial_i \left( a^{ij}(x') \partial_j w + b^i(x') w \right) + c^j(x') \partial_j w + d(x') w.
\]
We verify that $L$ is uniformly elliptic in $W$ and that it has bounded coefficients. Indeed, for each neighborhood $V \subset W$ of $x_0'$ there exists $K > 0$ so that $\sup_{t \in [0,1]} \| u^t \|_{C^1(\bar{V})} \leq K$, since $u_1, u_2 \in C^1(W)$. In particular, $a \in C^2$ implies that $\exists K_a > 0$ so that $\| a \|_{C^2(\bar{W} \times [-K, K])} \leq K_a$.

The assumed non-degeneracy (4.2) yields the uniform ellipticity, for $\xi' \in \mathbb{R}^{N-1}$ and $x' \in V$, $a^{ij}(x')\xi'_i\xi'_j \geq \alpha \| \xi' \|^2 / (1 + K^2)^{3/2}$. In addition, there exist constants $\Lambda(K, K), \nu(K, K)$ so

$$
\sup_{x' \in V} \sum_{i,j} (a^{ij}(x'))^2 \leq \Lambda, \quad \sup_{x' \in V} \sum_i |b^i(x')|^2 + \sum_j |c^j(x')|^2 + |d(x')| \leq \nu^2.
$$

By invoking the weak Harnack inequality [29, Thm. 8.18] on the non-negative supersolution $w$ of $Lw = 0$, we get the existence of $C > 0$ depending on $K, \alpha, \Lambda, \nu, N$, in such a way that

$$
\varepsilon_0^{-N/p}(\int_{B_{2\varepsilon_0}(x_0')} \| w \|^p)^{1/p} \leq C \inf_{B_{\varepsilon_0}(x_0')} w = 0,
$$

for every $p \in (1, N/(N-2))$ and $\varepsilon_0$ with $B_{4\varepsilon_0}(x_0') \subset V$. Therefore $w \equiv 0$ on $B_{\varepsilon_0}(x_0')$. \hfill \Box

### 4.3.2 Duality between weighted variations and mass of currents

The main tool in the proof of our desired Theorem 4.2 consists of a broader kind of maximum principle, for mass-minimizing currents established in [72]. With the purpose of applying this maximum principle to our setting (cf. Corollary 4.1) it will be then necessary to make the identification between the $a$-perimeter measure of a Caccioppoli set $E$, with the notion of mass of the co-dimension one rectifiable current $\partial [E]$, in the sense of Federer [27]. A reader well versed in geometric measure theory may consider this identification rather clear, however, for completeness and readability of the article we give a quick overview of some needed concepts in geometric measure theory.

Throughout, $M$ denotes an oriented complete smooth Riemannian manifold of dimension $n$ endowed with a $C^2$ metric $g$, $U$ denotes a (non-empty) open subset of $M$, and $l \in \{1, \ldots, N\}$. The $\alpha$-dimensional Hausdorff measure induced by the geodesic distance $d_g$ in $M$ will be written $\mathcal{H}^\alpha_g$. The space of smooth differential $l$-forms in $M$ is denoted by $\Omega^l(M)$. In particular, we are
interested in those forms that have compact support in $U$, for which we write

$$\mathcal{D}^l(U) := \{ \omega \in \Omega^l(M) : \text{supp}\, \omega \subseteq U \}.$$ 

An $l$-dimensional current in $U$ is defined as a continuous linear functional over $\mathcal{D}^l(U)$. The set of all $l$-dimensional currents over $U$ will denoted by $\mathcal{D}_l(U)$. Following [73, §26], we introduce a particularly useful class of currents of importance to us in the Riemannian setting, namely, the class integer multiplicity rectifiable currents (cf. [27]).

**Definition 4.3 (Integer multiplicity rectifiable current).** Let $U$ be an open set of $(M, g)$. If $T \in \mathcal{D}_l(U)$, we say $T$ is an integer multiplicity rectifiable current if it can be expressed as

$$T(\omega) = \int_N \langle w, \xi \rangle_x \, \theta(x) \, d\mathcal{H}^l(x), \quad \omega \in \mathcal{D}^l(U),$$

where $N$ is an $\mathcal{H}^l_g$-measurable countably $l$-rectifiable subset of $U$, $\theta$ is an integer valued function in $L^1_{\text{loc}}(N, \mathcal{H}^l_g)$, $\xi$ is a $\mathcal{H}^l_g$-measurable, oriented unit $l$-vector field on $N$, and $\langle \cdot, \cdot \rangle_x$ corresponds to the dual pairing between the spaces $\Lambda^l(T_x N)$ of $l$-covectors and $\Lambda_l(T_x N)$ of $l$-vectors, in the approximate tangent space $T_x N$.

In the case $T$ is as in (4.14), we write $T =: \tau(N, \theta, \xi)$ and call $\theta$ the multiplicity of $T$, $\xi$ the orientation of $T$. A central role in the maximum principle in [72] is played by currents induced by sets $E$, obtained by integration over $E$ of smooth $l$-forms in $M$ with compact support. That is,

**Definition 4.4 (Current induced by a set).** Let $E$ be a countably rectifiable subset of $M$ which is $\mathcal{H}^l_g$-measurable for some $l \in \{1, \ldots, N\}$. We define the $l$-dimensional current $[E]$, acting on $\eta \in \mathcal{D}^l(M)$ via

$$[E](\eta) := \int_E \eta,$$

where the integration is generally defined in the sense of [73, §11.1, §11.7].

We point out that (4.15) yields a sensible definition of $[E]$ even when the ambient space $M$ is not equipped with a metric $g$. Nonetheless, in the Riemannian setting it is convenient from the
point of view of geometric measure theory to rewrite this expression in terms of a measure arising from the metric of $M$. Thus, we have an alternative characterization for currents arising from appropriate sets, as given in

**Proposition 4.6.** For $E$ as in Definition 4.4, we have that $[E]$ is a multiplicity one rectifiable current, $[E] = \tau(E, 1, \xi)$, where $\xi$ is the oriented unit $l$-vector field on $E$ inherited from $M$. In other words,

$$[E](\eta) = \int_E \langle \eta, \xi \rangle_x \, d\mathcal{H}_g^l(x) \quad \text{for} \; \eta \in \mathcal{D}_l(M).$$

We now continue by recalling the notion of boundary of currents. Motivated by the classical Stoke’s theorem, we are led by (4.15) to quite generally define the boundary $\partial T$ of an $l$-current $T \in \mathcal{D}_l(M)$ by $\partial T(\omega) := T(d\omega)$ for $\omega \in \mathcal{D}^{l-1}(M)$. Finally we review the concept of mass of a current, in the sense of [73, §26.4]. Again motivated by the special case $T = [E]$ as in (4.15) we define the mass of the current $T$ with respect to the metric $g$, denoted $M_g(T)$, for $T \in \mathcal{D}_l(M)$ by

$$M_g(T) := \sup \left\{ T(\omega) : \omega \in \mathcal{D}_l(M), \sup_{x \in M} |\omega_x|^g \leq 1 \right\}, \quad (4.16)$$

(so that $M_g(T) = \mathcal{H}_g^m(E)$, in case $T = [E]$). More generally for any open set $U \subset M$ we define the mass of the current $T$ restricted to $U$ with respect to the metric $g$, by

$$M_{U,g}(T) := \sup \left\{ T(\omega) : \omega \in \mathcal{D}_l(M), \sup \omega \subset U, \sup_{x \in U} |\omega_x|^g \leq 1 \right\}, \quad (4.17)$$

where in (4.16)-(4.17) we adopt the convention where the norm of an $l$-covector $\omega_x$, $|\omega_x|^g$, is defined as the dual norm to $g$, that acts on the space of $l$-vectors.

We are ready to discuss the main result of this subsection, in which we identify the $a$-perimeter of a set $E \subset \mathbb{R}^N$ with the mass of the current $[E]$ in the ambient manifold $M = \mathbb{R}^N$, with respect to a certain metric $g$ depending on the weight function $a$. This will allow us to invoke the maximum principle in [72] to our development.
Theorem 4.4. Consider \( \mathbb{R}^N \) endowed with the metric \( g = \sigma \bar{g} \), conformal to the standard Euclidean \( \bar{g}_x = \delta_{ij}dx^i dx^j \) with \( \sigma \neq 0 \). If \( E \) is a Caccioppoli set that is furthermore an \( N \)-rectifiable Borel set, then there holds
\[
M_{U,\sigma g}(\partial [E]) = \int_{U} \sigma^{(N-1)/2}(x) d|\partial X|_g(x).
\]
for any connected open set \( U \subset \mathbb{R}^N \). In particular, the choice \( \sigma = 2/(N-1) \) yields
\[
M_{U,\sigma^{2/(N-1)}g}(\partial [E]) = \mathcal{P}(E, U).
\]

Proof of Theorem 4.4. Clearly \( [\partial/\partial x^1, \ldots, \partial/\partial x^N] \) is a global orientation of the ambient manifold \( \mathbb{R}^N \), where \((x^1, \ldots, x^N)\) denote the standard coordinates. The countably \( N \)-rectifiable set \( E \) is \( \mathcal{H}^N \)-measurable since it is Borel in \( \mathbb{R}^N \), hence it induces the \( N \)-current \([E] := \tau(E, 1, \xi)\) as in Proposition 4.6 (with \( l = N \)), where \( \xi \) is the unit standard orientation in \( E \) induced from \( \mathbb{R}^N \).

According to the definitions of mass in (4.17) and of boundary current, \( M_{U,g}(\partial [E]) \) is the supremum of \([E](d\omega)\) for any \( \omega \in \mathcal{D}^{N-1}(\mathbb{R}^N) \) with \( \text{supp}\omega \subset U \) and \( \sup_{x \in U} |\omega(x)|_g \leq 1 \). On the other hand, it is known that the Hodge star operator \( * : \Omega^1(U) \to \Omega^{N-1}(U) \) is a linear isometry, see e.g. [60, Prop. 4.7]. Thus, \( \omega \in \Omega^{N-1}(U) \) can be written \( \omega = *\Upsilon \) for \( \Upsilon \in \Omega^1(U) \), and furthermore 1-forms admit a vector field representation by means of the musical isomorphism:
\( \Upsilon = X^\flat \), which is an isometry by construction of the flat operator \( \flat : C^\infty(U; \mathbb{R}^N) \to \Omega^1(U) \). Let us now remark that \( \text{supp} X \subset U \) iff \( \text{supp}(*X^\flat) \subset U \). Indeed, we can write explicitly in coordinates
\[
*X^\flat = * \left( \sum_{j=1}^N \left( \sum_{i=1}^N g_{ij} X^i \right) \frac{\partial}{\partial x_j} \right) = \sum_{j=1}^N \left( \sum_{i=1}^N g_{ij} X^i \right) (-1)^{j-1} dx^1 \wedge \ldots \wedge \hat{dx^j} \wedge \ldots \wedge dx^N.
\]
In light of these observations, it follows that \(|(*X^\flat)(x)|_g = |X^\flat(x)|_g = |X(x)|_g\), and therefore
\[
M_{U,g}(\partial [E]) = \sup \left\{ [E](d*X^\flat) : X \in C^\infty(U; \mathbb{R}^N), \text{supp} X \subset U, \sup_{x \in U} |X(x)|_g \leq 1 \right\}.
\] (4.18)
In addition, it is known that \( *X^\flat = dV_g(X, \cdot) \), so now the definition of divergence in Riemannian manifolds implies that \( d*X^\flat = (\text{div}_g X) dV_g \). Also, the volume form \( dV_g \) satisfies that \( \langle dV_g, \xi \rangle \equiv 1 \) for any unit \( n \)-vector field \( \xi \). Consequently,
\[
[E](d*X^\flat) = \int_E (\langle \text{div}_g X \rangle dV_g, \xi)_x d\mathcal{H}^N_g(x) = \int_E \text{div}_g X d\mathcal{H}^N_g.
\]
Here $d\mathcal{H}^N_g$ denotes the Riemannian measure. As $E$ is countably $N$-rectifiable, in the study the last integral we can assume with no loss of generality that $E$ is an $N$-dimensional embedded $C^1$-submanifold of $\mathbb{R}^N$, cf. [73, §11.1]. In this case, let us note the pullback metric in $E$ is again $g$, and that in the Euclidean ambient spaces there is no need to use local parametrizations and partitions of unity to compute the integration with respect to the Riemannian measure in $E$. From all of this, and the expression in coordinates of the divergence operator of $X = \sum_j X^j (\partial/\partial x^j) \in C^\infty_c(U; \mathbb{R}^N)$, we simply obtain

$$[E](d \ast X^\flat) = \int_E \sqrt{\det g} \text{div}_g X \, dx$$

$$= \int_E \sum_{j=1}^N \frac{\partial}{\partial x^j} (\sqrt{\det g} X^j) \, dx$$

$$= \int_{\mathbb{R}^N} \chi_E \text{div}(\sqrt{\det g} X) \, dx,$$

where $\text{div}(\cdot) = \text{div}_g(\cdot)$ stands for the divergence in Euclidean coordinates. Defining now the vector field $Y := \sqrt{\det g} X \in C^\infty_c(U; \mathbb{R}^N)$, we get from $g = a^\sigma g$ that $\sqrt{\det g(x)} = a^{N\sigma/2}(x)$, and consequently

$$|X|_g = \sqrt{g_x(X,X)} = a^{\sigma/2} |X|_g = a^{(1-N)\sigma/2}(a^{N\sigma/2}|X|_g) = a^{(1-N)\sigma/2} |Y|_g.$$

The computation of the mass $M_{U,g}(\partial[E])$ in (4.18) has been shown to be equivalent to

$$\sup \left\{ \int_{\mathbb{R}^N} \chi_E \text{div} Y \, dx : Y \in C^\infty_c(U; \mathbb{R}^N), \sup_{x \in U} a^{-(N-1)\sigma/2}(x)|Y(x)|_g \leq 1 \right\}. \quad (4.19)$$

Finally, we observe that $a^{-(N-1)\sigma/2}(x)|Y(x)|_g = \varphi^0_\sigma(x, Y(x))$, where $\varphi^0_\sigma(x, \cdot)$ is the polar (or dual) of the norm $\varphi_\sigma(x, \cdot) := a^{(N-1)\sigma/2}(\cdot)|_g$. This fact along with definition (4.4) allows us to identify the supremum in (4.19) as an inhomogeneous variation of $\chi_E$ in $U \subset \mathbb{R}^N$,

$$\int_U a^{(N-1)\sigma/2}(x)|D\chi_E|,$$

thus completing the proof of Theorem 4.4. \qed
Remark 4.1. The singularity estimates (4.11) mentioned in 4.2 follow directly from the regularity theory in [70, 71] for co-dimension one rectifiable currents minimizing parametric elliptic functionals, by choosing $F(T) = \int_{\mathbb{R}^N} F(x, \nu^T(x)) d\|T\|$ with $F(x, p) = a(x)|p|$ (following their notation) and provided $a \in C^3(\Omega)$ and is bounded away from zero. However, focusing simply in the question of existence of tangent cones at every point on the support of a mass-minimizing boundary current $T = \partial \mathbb{E}$ in the Riemannian manifold setting, one can just invoke the identification in Theorem 4.4 above. Indeed, one can apply what are now standard techniques (cf. Federer [27] or Simon [73]) in the regularity theory of integral mass-minimizing currents in the Euclidean space $(\mathbb{R}^N, \bar{g})$ with $\bar{g}(x) = \delta_{ij} dx^i dx^j$. Given $x_0 \in \text{supp} T$ on a Riemannian manifold $(M, g)$ we let $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$ be normal coordinates for $M$ near $x_0$ with the origin $x = 0$ corresponding to $x_0$ and with $T_{x_0} M$ identified with $\mathbb{R}^N$ via these coordinates (possible for $g \in C^2$). Then $g(x) = g_{ij} dx^i dx^j$ with $g_{ij}(0) = \delta_{ij}$ and $\partial g_{ij}/\partial x^k(0) = 0$ for all $i, j, k$. We can take homotheties $T_\lambda := (\lambda^{-1})_# T$ for $\lambda > 0$ in terms of these local coordinates, and $T_\lambda$ is mass-minimizing relative to the metric $g_{ij}(\lambda x) dx^i dx^j$. In light of the approximation for $g$, Euclidean density estimates around $x = 0$ translate into estimates for the mass ratio of the area in metric balls around $x_0$. A monotonicity formula for the density function is then available, which combined with the compactness theorem for locally rectifiable currents (in the Riemannian setting) [59, Thm. 5.5] show that for any $\lambda_j \to 0^+$ the sequence $T_{\lambda_j}$ has a subsequence converging weakly to some current $C$ (i.e. the tangent cone at $x_0$), invariant under large homotheties. A detailed account of this argument is given in [58, p.5044]. A further recollection of results for currents in the Riemannian setting can be found in [36].

4.3.3 Proof of strict maximum principle Theorem 4.2

The argument of this result in the Euclidean case (i.e. $a(x) \equiv 1$) is given [80, Thm 2.2]. In what follows, we indicate how their proof goes through as well in our inhomogeneous setting given by a
weight function $\alpha(x)$.

As in the statement of the theorem, consider sets $E_1$ and $E_2$ minimizing the $\alpha$-perimeter in an open set $U \subset \mathbb{R}^N$, and we assume that $x \in \partial E_1 \cap \partial E_2 \cap U$. Let us select $r > 0$ small so that $B_r(x) \subset \subset U$ and observe that, in light of $\overline{\partial^* E_j} = \partial E_j$ (see e.g. [30, §4]), it follows that $x$ is in the closure of $(\partial^* E_j) \cap B_r(x)$, for both $j = 1, 2$.

Also, according to Definition 4.4, each connected component $S_{j,m}$ of $(\partial^* E_j) \cap B_r(x)$ for each $j = 1, 2$ and $m \geq 0$, can be considered as an $(N - 1)$-dimensional multiplicity one rectifiable current $T_{j,m} = \left[ [S_{j,m}] \right] \ll_{B_r(x)}$ in the ambient Riemannian manifold $\mathbb{R}^N$ endowed with the metric $g(x) = \alpha^{2/(N-1)}(x)\delta_{ij}dx^idx^j$. The choice of the power $2/(N - 1)$ in the weight function $\alpha$ guarantees that $M_{B_r(x),g}(\partial [E_j]) = \mathcal{P}_\alpha(E_j, B_r(x))$, according to Theorem 4.4. In particular, since each $\partial E_j$ minimizes the $\alpha$-area in $B_r(x)$, then every $S_{j,m}$ is in fact a mass minimizing current for the metric $g$ considered as above. This condition together with the non-degeneracy (4.2) of $\alpha$ implies that any such component $S_{j,m}$ which intersects $B_{\rho/2}(x)$, for $\rho > 0$ small, satisfies furthermore $M_{B_{\rho}(x),g}(S_{j,m}) \geq c\rho^N$ (see e.g. [27, §5.1.6]), so at most finitely many components of $(\partial^* E_j) \cap B_{\rho}(x)$ can intersect $B_{\rho/2}(x)$.

This argument shows as a matter of fact that for each $j$, $x$ lies in the closure of some connected component of $(\partial^* E_j) \cap B_r(x)$. Let us write $x \in \bar{C}_1 \cap \bar{C}_2$ where $C_j = S_{j,m_j}$ for some $m_1, m_2 \geq 0$. Moreover, Theorem 4.3 yields that $x$ is in the closure of no other component of $(\partial^* E_j) \cap B_r(x)$, so the numbers $m_1, m_2$ are uniquely determined. If $C_1 \cap C_2 = \emptyset$, it follows from the proof of Theorem 4.3 that there exist sets $F_j \subset B_r(x)$ with $\partial F_j = C_j$ and $F_1 \subset F_2$. Then from the Corollary 4.1 we conclude $\bar{C}_1 = \bar{C}_2$. Because there are only finitely many components of $(\partial^* E_j) \cap B_r(x)$ that can intersect any compact subset of $B_r(x)$, it follows that there exists $\rho > 0$ so that $(\bar{C}_j) \cap B_{\rho}(x) = (\partial E_j) \cap B_{\rho}(x)$, $j = 1, 2$. Thus, in the case $C_1 \cap C_2 = \emptyset$ we get the desired result since $(\partial E_1) \cap B_{\rho}(x) = (\partial E_2) \cap B_{\rho}(x)$, whereas in the case $C_1 \cap C_2 \neq \emptyset$ we use the fact that $E_1 \subset E_2$ to conclude that $C_1$ lies locally on one side of $C_2$ near each point of $C_1$. Now the Proposition 4.5
implies that $C_1 = C_2$. Hence $\bar{C}_1 = \bar{C}_2$ and this, as above, concludes the proof of Theorem 4.2. □

4.4 A geometric variational problem

In this section we introduce an auxiliary variational problem for subsets of $\mathbb{R}^N$, which will be the cornerstone in the construction of a solution $u$ to the original (aLGP). We follow the outline of the strategy introduced in [80].

Let us write henceforth $[a, b] := g(\partial \Omega)$. Observe that $g \in C(\partial \Omega)$ admits a continuous extension on the complement of $\Omega$,

$$G \in BV(\mathbb{R}^N \setminus \bar{\Omega}) \cap C(\mathbb{R}^N \setminus \Omega), \quad G = g \text{ on } \partial \Omega.$$  

In fact, as shown in [30, Thm. 2.16], we can require that $\text{supp}(G) \subset B_R(0)$ with $R$ large so that $\Omega \subset B_R(0)$. Next, we introduce sets that will ensure our constructed solution satisfies the Dirichlet boundary condition $u = g$ on $\partial \Omega$. For each $t \in [a, b]$, we let

$$\mathcal{L}_t := \{x \in (\mathbb{R}^N \setminus \Omega) : G(x) \geq t\}. \quad (4.20)$$

The weighted co-area formula (Proposition 4.2) and the fact $G \in BV(\mathbb{R}^N \setminus \bar{\Omega})$ both imply that $\mathcal{P}_a(\mathcal{L}_t, \mathbb{R}^N \setminus \bar{\Omega}) < \infty$, for a.e. $t \in [a, b]$. However, the construction of the extension above can be made so that we also have $\mathcal{P}_a(\mathcal{L}_t, \mathbb{R}^N \setminus \bar{\Omega}) < \infty$ for all $t \in [a, b]$, cf. [30, Thm. 2.16].

We remind the reader that we employ our convention (4.7) in defining $\mathcal{L}_t$. Using (4.9) and $\mathcal{H}^{N-1}(\partial \Omega) < \infty$, along with the property $\max_{\partial \Omega} a < \infty$, we deduce that for any $t \in [a, b]

$$\mathcal{P}_a(\mathcal{L}_t, \mathbb{R}^N) = \mathcal{P}_a(\mathcal{L}_t, \mathbb{R}^N \setminus \bar{\Omega}) + \mathcal{H}^{N-1}(a((\partial_M \mathcal{L}_t) \cap (\partial \Omega))) < \infty. \quad (4.21)$$

For each $t \in [a, b]$, consider the variational problems

$$\inf\{\mathcal{P}_a(E, \mathbb{R}^N) : E \setminus \bar{\Omega} = \mathcal{L}_t \setminus \bar{\Omega}\}, \quad (\star_t)$$

$$\sup\{|E| : E \text{ is a solution of } (\star_t)\}. \quad (**_t)$$

Let us first observe that
Proposition 4.7. The problem \((**t)\) has a solution, for every \(t \in [a, b]\).

Proof of Proposition 4.7. We argue the existence of a solution to \((\star t)\), because this shows on the one hand that the admissible set in \((**t)\) is non-empty, but also the existence of a solution to \((**t)\) can be derived along the same lines as for \((\star t)\).

Let us write \(m_t\) for the infimum value in \((\star t)\). Observe \(m_t < +\infty\), because \(L_t\) is admissible for \((\star t)\) with \(P_a(L_t, \mathbb{R}^N) < +\infty\) due to (4.21). Recall that \(\Omega \subset \subset B_R\) for \(R\) large enough, and let \(R\) represent this value throughout. Consider the auxiliary problem
\[
\tilde{m}_t := \inf \{ P_a(F, B_R) : F \subset B_R, F \cap (B_R \setminus \overline{\Omega}) = L_t \cap (B_R \setminus \overline{\Omega}) \}.
\] (4.22)

As before, it is easy to see that \(\tilde{m}_t < +\infty\). Let \(\{F_j\}\) be a minimizing sequence for (4.22), so \(P_a(F_j, B_R) \to \tilde{m}_t\) as \(j \to \infty\) and \(F_j \subset B_R, F_j \cap (B_R \setminus \overline{\Omega}) = L_t \cap (B_R \setminus \overline{\Omega})\) for all \(j\). The non-degeneracy (4.2) of the weight function, and the representation formula (4.9) of the \(a\)-perimeter yield
\[
\|\chi_{F_j}\|_{BV(B_R)} = |F_j| + \int_{B_R} |D\chi_{F_j}| \leq |B_R| + \frac{1}{\alpha} \int_{B_R} a(x) d|D\chi_{F_j}|
\]
\[
\leq |B_R| + (\tilde{m}_t + 1)/\alpha < \infty,
\]
provided \(j\) is large enough. Then, recalling the compact embedding \(BV(B_R) \hookrightarrow L^1(B_R)\) (see e.g. [30]), we see that there exist a subsequence (denoted in the same way) and a set \(\mathcal{F}_t \subset \mathbb{R}^N\) so that \(\chi_{F_j} \rightharpoonup \chi_{\mathcal{F}_t}\) in \(L^1(B_R)\), as \(j \to \infty\). This fact combined with the l.s.c. property of the \(a\)-variation (Proposition 4.4), shows that
\[
|\mathcal{F}_t| \leq |B_R|,
\]
\[
\mathcal{P}_a(\mathcal{F}_t, B_R) \leq \liminf_{j \to \infty} \mathcal{P}_a(F_j, B_R) = \tilde{m}_t.
\]
We will have shown that \(\mathcal{F}_t\) solves (4.22), once we argue that \(\mathcal{F}_t\) is admissible for this problem. In view of our convention (4.7), the admissibility follows if
\[
|(\mathcal{F}_t \cap (B_R \setminus \overline{\Omega})) \Delta (L_t \cap (B_R \setminus \overline{\Omega}))| = 0.
\]
Since each $F_j$ is admissible for ($\star_t$), the aforementioned $L^1$-convergence implies
\[
|\mathcal{F}_t \cap (B_R \setminus \bar{\Omega})\Delta (\mathcal{L}_t \cap (B_R \setminus \bar{\Omega}))| = |(\mathcal{F}_t \Delta F_j) \cap (B_R \setminus \bar{\Omega})| \leq |(\mathcal{F}_t \Delta F_j) \cap B_R| = \int_{B_R} |\chi_{F_j} - \chi_{\mathcal{F}_t}| \, dx \xrightarrow{j \to \infty} 0.
\]

Now that we established the existence of a minimizer for the auxiliary problem, we easily get the one for the extended problem ($\star_t$). Indeed, we put $\mathcal{F}^*_t := (\mathcal{L}_t \setminus B_R) \cup \mathcal{F}_t$ and notice that $\mathcal{F}_t = (\mathcal{L}_t \cap (B_R \setminus \bar{\Omega})) \cup (\mathcal{F}_t \cap \bar{\Omega})$ shows that this set can be equivalently written
\[
\mathcal{F}^*_t = (\mathcal{L}_t \setminus \bar{\Omega}) \cup (\mathcal{F}_t \cap \bar{\Omega}).
\]

In particular, any competitor $E$ of ($\star_t$) satisfies
\[
\mathcal{F}^*_t \Delta E = \mathcal{F}^*_t \Delta \{(\mathcal{L}_t \setminus \bar{\Omega}) \cup (E \cap \bar{\Omega})\} = (\mathcal{F}_t \cap \bar{\Omega}) \Delta (E \cap \bar{\Omega}) \subset \subset B_R,
\]
thus implying the inequality
\[
\mathcal{P}_d(E, \mathbb{R}^N) - \mathcal{P}_d(\mathcal{F}^*_t, \mathbb{R}^N) = \mathcal{P}_d(E, B_R) - \mathcal{P}_d(\mathcal{F}^*_t, B_R) = \mathcal{P}_d(E \cap B_R, B_R) - \mathcal{P}_d(\mathcal{F}_t, B_R) \geq 0,
\]
where the last inequality comes from the fact that $E \cap B_R$ is admissible for (4.22):
\[
(E \cap B_R) \cap (B_R \setminus \bar{\Omega}) = (E \setminus \bar{\Omega}) \cap B_R = (\mathcal{L}_t \setminus \bar{\Omega}) \cap B_R = \mathcal{L}_t \cap (B_R \setminus \bar{\Omega}).
\]

In the remainder of this chapter we write $E_t$ for a solution to ($\star \star_t$); a well defined object in light of Proposition 4.7. Let us remark that our convention (4.7) ensures $E_t \setminus \bar{\Omega} = \mathcal{L}_t \setminus \bar{\Omega}$; and moreover, we observe that $\mathcal{L}_t$ need not be a closed set.

4.5 Construction of a minimizer in main Theorem 4.1

In this section we study two crucial properties of the collection of sets $\{E_t : t \in [a, b]\}$. These are going to allow us to construct a solution $u$ to $(a\text{LGP})$ by means of relating, up to an $\mathcal{H}^N$-measure
zero set, the superlevel set \( \{ u \geq t \} \) with \( E_t \cap \bar{\Omega} \), for almost all \( t \in [a, b] \). This construction will be carried out for bounded Lipschitz domains \( \Omega \) with connected boundary, which in addition satisfy the barrier condition (4.5) stated in the introduction.

A crucial step in our development is the following property of \( E_t \), the solution to (\( \star \star t \))

**Lemma 4.1 (Boundary values).** Suppose \( \partial \Omega \) satisfies the barrier condition with respect to the weight function \( a \). Then,

\[
\partial E_t \cap \partial \Omega \subset \{ g = t \} \text{ for } t \in [a, b].
\]

**Proof of Lemma 4.1.** Although we roughly follow the same line of argumentation as that found in [80], we make no use here of the PDE techniques invoked by those authors to reach a contradiction. Instead, we study topological properties of sets whose boundaries minimize the \( a \)-area, while using the barrier condition (4.5).

Arguing by contradiction, let us suppose there exists a point \( x_0 \in \partial E_t \cap \partial \Omega \) with \( g(x_0) < t \), and let \( \varepsilon_0 = \varepsilon_0(x_0) \) be the constant appearing in the barrier condition at \( x_0 \). It will be convenient to write \( B_\varepsilon := B_\varepsilon(x_0) \) for this proof only, to simplify the notation.

Let us observe that the continuity of \( G \) together with the condition \( g(x_0) < t \) guarantees

\[
\mathcal{L}_t \cap B_\varepsilon = \emptyset,
\]

provided \( 0 < \varepsilon < \varepsilon_0 \) is taken small enough. We now fix such an \( \varepsilon \). Since \( E_t \) is a solution of (\( \star t \)), it then follows that

\[
E_t \cap B_\varepsilon \subset \bar{\Omega}.
\]

Before continuing, we remark that existence of a minimizer \( V_* \) of the variational problem (4.5) can be established using the direct method in the calculus of variations, in the spirit of the proof of Proposition 4.7.

Our main claim is that \( E_t \) is not a minimizer of (\( \star t \)), thus contradicting the definition of the set \( E_t \). This will be achieved by proving that \( \mathcal{P}_d(E_t, \mathbb{R}^N) > \mathcal{P}_d(\mathcal{E}_*, \mathbb{R}^N) \), where \( \mathcal{E}_* := (E_t \cap V_*) \cap \)
$B_\varepsilon) \cup (E_t \setminus B_\varepsilon)$. To see this, let us first note from (4.23)-(4.24) that $\varepsilon_*$ is a competitor in $(*_t)$. Moreover, since $E_t = \varepsilon_*$ in $\mathbb{R}^N \setminus B_\varepsilon$ we can use the characterization of the $a$-perimeter measure (4.9), to obtain

$$
\mathcal{P}_a(E_t, \mathbb{R}^N) - \mathcal{P}_a(\varepsilon_*, \mathbb{R}^N) = \mathcal{P}_a(E_t, B_\varepsilon) - \mathcal{P}_a(\varepsilon_*, B_\varepsilon)
$$

(4.25)

where

$$
\mathcal{H}^{N-1} a(\partial^* E_t \cap B_\varepsilon) = \mathcal{H}^{N-1} a(\partial^* E_t \cap V_* \cap B_\varepsilon)
+ \mathcal{H}^{N-1} a((\partial^* E_t \setminus V_*) \cap B_\varepsilon),
$$

$$
\mathcal{H}^{N-1} a(\partial^* (E_t \cap V_* \setminus B_\varepsilon) = \mathcal{H}^{N-1} a(\partial^* E_t \cap V_* \cap B_\varepsilon)
+ \mathcal{H}^{N-1} a(\partial^* V_* \cap E_t \cap B_\varepsilon),
$$

see Figure 4.1 (a) below. Applying these identities to (4.25) we obtain

$$
\mathcal{P}_a(E_t, \mathbb{R}^N) - \mathcal{P}_a(\varepsilon_*, \mathbb{R}^N) = \mathcal{H}^{N-1} a((\partial^* E_t \setminus V_*) \cap B_\varepsilon) - \mathcal{H}^{N-1} a(\partial^* V_* \cap E_t \cap B_\varepsilon). \quad (4.26)
$$

![Figure 4.1](image-url)

Figure 4.1: Sketch of contradiction in the cases: (a) $g(x_0) < t$, and (b) $g(x_0) > t$.

On the other hand, let us observe that the set $\mathcal{V} := V_* \cup (E_t \cap \Omega)$ is admissible for problem (4.5), and furthermore (4.24) shows that $E_t \cap B_\varepsilon = (E_t \cap \Omega) \cap B_\varepsilon \mathcal{H}^N$-a.e. Consequently

$$
\mathcal{V} = ([V_* \cup E_t] \cap B_\varepsilon) \cup (V_\setminus B_\varepsilon), \quad (4.27)
$$
in light of our convention (4.7). In particular, the statement $x_0 \in \partial E_t$ can be made more precise by means of the barrier condition of $\partial \Omega$ for the minimizer $V_*$ and (4.27), to read now $x_0 \in (\partial E_t \setminus \bar{V}_*) \subset \partial \mathcal{V}$. Put another way, $x_0 \in \partial \mathcal{V} \cap \partial \Omega \cap B_\varepsilon$, so by virtue of the barrier condition once again we see that $\mathcal{V}$ cannot be a minimizer of (18). Hence, by the characterization (4.9) of the weighted perimeter and the minimality of $V_*$, we derive

$$0 < \mathcal{P}_a(\mathcal{V}, \mathbb{R}^N) - \mathcal{P}_a(V_*, \mathbb{R}^N) = \mathcal{P}_a(\mathcal{V}, B_\varepsilon) - \mathcal{P}_a(V_*, B_\varepsilon)$$

$$= \mathcal{H}^{N-1} \bigcap a(\partial^*(V_* \cup E_t) \cap B_\varepsilon) - \mathcal{H}^{N-1} \bigcap a(\partial^* V_* \cap B_\varepsilon),$$

where we used that $\mathcal{V} = V_*$ in $\mathbb{R}^n \setminus B_\varepsilon$. This inequality can be exploited in light of the identities below:

$$\mathcal{H}^{N-1} \bigcap a(\partial^*(V_* \cup E_t) \cap B_\varepsilon) = \mathcal{H}^{N-1} \bigcap a((\partial^* E_t \setminus V_*) \cap B_\varepsilon)$$

$$+ \mathcal{H}^{N-1} \bigcap a((\partial^* V_* \setminus E_t) \cap B_\varepsilon),$$

$$\mathcal{H}^{N-1} \bigcap a(\partial^* V_* \cap B_\varepsilon) = \mathcal{H}^{N-1} \bigcap a((\partial^* E_t \setminus \bar{V}_*) \cap B_\varepsilon)$$

$$+ \mathcal{H}^{N-1} \bigcap a((\partial^* \bar{V}_* \setminus E_t) \cap B_\varepsilon),$$

to simply get

$$\mathcal{H}^{N-1} \bigcap a(\partial^* V_* \cap E_t \cap B_\varepsilon) < \mathcal{H}^{N-1} \bigcap a((\partial^* E_t \setminus V_*) \cap B_\varepsilon). \quad (4.28)$$

It immediately follows from (4.26) and (4.28) that

$$\mathcal{P}_a(E_t, \mathbb{R}^N) - \mathcal{P}_a(\mathcal{E}_*, \mathbb{R}^N) > 0,$$

thus finishing the proof of the contradiction argument in the case $g(x_0) < t$.

The other case where $g(x_0) > t$ is argued again by contradicting the minimality of $E_t$ in (⋆t), nonetheless, for the sake of completeness we briefly discuss the proof. Indeed, the continuity of the extension $G$ of $g$ shows

$$(B_\varepsilon \setminus \bar{\Omega}) \subset E_t \cap B_\varepsilon \quad \text{if and only if} \quad (B_\varepsilon \setminus \bar{E}_t) \subset \bar{\Omega} \cap B_\varepsilon. \quad (4.29)$$

The claim is that $\mathcal{P}_a(E_t, \mathbb{R}^N) > \mathcal{P}_a(\mathcal{E}_*, \mathbb{R}^N)$, if we now take $\mathcal{E}_* := (E_t \cup (\Omega \setminus V_*)) \cap B_\varepsilon \cup (E_t \setminus B_\varepsilon)$. 

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Just as in the previous case we analyze the difference
\[
\mathcal{P}_a(E_t, \mathbb{R}^N) - \mathcal{P}_a(E_s, \mathbb{R}^N) = \mathcal{H}^{N-1} \bigcap a(\partial^* E_t \cap B_{\varepsilon}) - \mathcal{H}^{N-1} \bigcap a(\partial^* (E_t \cup (\Omega \setminus V_*)) \cap B_{\varepsilon})
\]
\[
= \mathcal{H}^{N-1} \bigcap a((\partial^* E_t \setminus V_*) \cap B_{\varepsilon}) - \mathcal{H}^{N-1} \bigcap a((\partial^* V_* \setminus E_t) \cap B_{\varepsilon}),
\]
(4.30)
see Figure 1(b) above.

Let us consider now a new auxiliary set \( V_* := V_* \cup (\Omega \setminus E_t) \), and perform a similar analysis as before. Noting \( x_0 \in \partial E_t \), from (4.29) we get \( x_0 \in \partial (\Omega \setminus E_t) \) which, in addition to \( x_0 \notin \bar{V}_* \) (by the barrier condition), implies that \( x_0 \in \partial V_* \). Since \( V_* \) is admissible in \((\star_t)\), the barrier condition once again yields
\[
\mathcal{P}_a(V_*, \mathbb{R}^N) - \mathcal{P}_a(V_*, \mathbb{R}^N) = \mathcal{H}^{N-1} \bigcap a((\partial^* V_* \setminus E_t) \cap B_{\varepsilon}) - \mathcal{H}^{N-1} \bigcap a((\partial^* V_* \setminus E_t) \cap B_{\varepsilon})
\]
\[\geq 0,\]
leading to the desired conclusion, in view of (4.30). The proof of Lemma 4.1 is now complete. \( \square \)

Continuing the study of the family \( \{E_t : t \in [a,b]\} \), we give a basic geometric description on how they are positioned inside of the domain \( \Omega \).

**Lemma 4.2.** Suppose \( \partial \Omega \) satisfies the barrier condition with respect to \( a \). Then, for any \( s, t \in [a,b] \) with \( s < t \), there holds
\[
E_t \subset\subset E_s.
\]

**Proof of Lemma 4.2.** The containment \( E_t \subset E_s \) follows from the same argument as in [80]. Let us observe \( E_s \cap E_t \) is a competitor with \( E_t \) in \((\star_t)\),
\[
(E_s \cap E_t) \setminus \bar{\Omega} = (E_s \setminus \bar{\Omega}) \cap (E_t \setminus \bar{\Omega})
\]
\[
= (\mathcal{L}_s \setminus \bar{\Omega}) \cap (\mathcal{L}_t \setminus \bar{\Omega}) = \mathcal{L}_t \setminus \bar{\Omega}.
\]
In a similar fashion, it can be readily seen \( E_s \cup E_t \) is a competitor with \( E_s \) in \((\star_s)\). It follows
\[
\mathcal{P}_a(E_s \cap E_t, \mathbb{R}^N) \geq \mathcal{P}_a(E_t, \mathbb{R}^N) \quad \text{and} \quad \mathcal{P}_a(E_s \cup E_t, \mathbb{R}^N) \geq \mathcal{P}_a(E_s, \mathbb{R}^N).
\]
As the \( \alpha \)-perimeter satisfies Proposition 4.3, the above inequalities imply \( \mathcal{P}_\alpha(E_s \cup E_t, \mathbb{R}^N) = \mathcal{P}_\alpha(E_s, \mathbb{R}^N) \). On the other hand, \( E_s \) solves the problem \((\ast\ast_s)\) thus yielding \( |E_s \cup E_t| = |E_s| \). We conclude \( |E_t \setminus E_s| = 0 \), which in view of our convention (4.7) then yields

\[
E_t \subset E_s. \tag{4.31}
\]

It remains to show that this containment is actually strict. This method uses topological arguments along with techniques from geometric measure theory, and is an adaptation from the proof of this lemma in [80]. Let us start noting

\[
E_t \setminus \Omega = L_t \setminus \tilde{\Omega} \subset L_s \setminus \tilde{\Omega} = E_s \setminus \tilde{\Omega}, \tag{4.32}
\]

relative to the topology on \( \Omega^c \). In addition, since \( s < t \) we observe that Lemma 4.1 implies

\[
\partial E_t \cap \partial E_s \cap \partial \Omega = \emptyset. \tag{4.33}
\]

In consideration of (4.31)-(4.32)-(4.33) we will prove the statement of this Lemma by showing

\[
S := \partial E_s \cap \partial E_t \cap \Omega = \emptyset.
\]

For this purpose let us assume on the contrary that \( S \neq \emptyset \). The goal of the remainder of our proof is to verify the points below:

(i) \( S \) consists of the connected components of \( \partial E_t \) that do not intersect \( \partial \Omega \).

(ii) If \( S' \) denote any connected component of \( \text{reg}(S) \), then \( \tilde{S}' \) has to intersect \( \partial \Omega \).

Using the density of \( \text{reg}(\partial E_t) \) in \( \partial E_t \) (see (4.11) in §4.2), we immediate conclude from (i)-(ii) that \( S \) must be empty, thus reaching a contradiction.

To argue (i), first note that \( S \) open relative to \( \partial E_t \), for if \( x \in S \), since \( E_t \subset E_s \) and both \( \partial E_t \), \( \partial E_s \) are \( \alpha \)-area minimizing in \( \Omega \), we can apply the maximum principle in §4.3, Theorem 4.2, to conclude that \( \partial E_t \) and \( \partial E_s \) agree on a neighborhood of \( x \). On the other hand, \( S \) is clearly closed relative to \( \partial E_t \), so from (4.33) we get that every connected component of \( S \) is disjoint from \( \partial \Omega \).
The proof of (ii) is based on the fact that \( \partial E_t \) is \( \alpha \)-perimeter minimizing in \( \Omega \). This is a rather general fact as given in the following

**Lemma 4.3** ([39, Lem. 4.5]). *Let \( \Omega \) be a bounded Lipschitz domain with connected boundary, and assume \( E \subset \mathbb{R}^N \) is \( \alpha \)-perimeter minimizing in \( \Omega \). If \( R \) is a connected component of \( \text{reg}(\partial E) \cap \Omega \), then \( \overline{R} \cap \partial \Omega \neq \emptyset \).*

The proof of Lemma 4.3 is based on topological arguments in geometric measure theory. A full proof can be found in [39], where the authors argue by contradiction that if a set \( E \) is \( \alpha \)-perimeter minimizing and admits some component of \( \text{reg}(\partial E) \) whose closure is disjoint from \( \partial \Omega \), then \( E \) could have not been \( \alpha \)-perimeter minimizing in the beginning.

### 4.6 Proof of the main Theorem 4.1

We are now in position to build up a *continuous* solution to the weighted least gradient problem \((\alpha\text{LGP})\). For \( t \in [a, b] \), we define

\[
A_t := (E_t \cap \Omega).
\]

Let us observe that \( A_t \cap \Omega = E_t \cap \Omega \), because of the convention (4.7) in view of \( \partial E_t = \partial_M E_t \). This follows because each point in \( \partial E_t \) is either a regular point of \( \partial E_t \), or a point at which the tangent cone exists. Also, from topological considerations and the help of Lemma 4.1, there also hold for \( t \in [a, b] \) that

\[
\{g > t\} \subset (E_t)^i \cap \partial \Omega \subset A_t \cap \partial \Omega,
\]

\[
\overline{\{g > t\}} \subset A_t \cap \partial \Omega \subset [(E_t)^i \cup \partial E_t] \cap \partial \Omega \subset \{g \geq t\}.
\]

Finally, we observe that for any \( s < t \) with \( s, t \in [a, b] \)

\[
A_t \subset \subset A_s,
\]
relative to $\bar{\Omega}$. Indeed, topological considerations show that for a.e. \( t \), $\partial_{\bar{\Omega}} A_t \subset \partial E_t$, where $\partial_{\bar{\Omega}}$ denotes the topological boundary relative to the subspace topology of $\bar{\Omega}$. The validity of (4.36) is then a mere consequence of Lemma 4.1 and Lemma 4.2 combined.

The definition of our candidate for the solution, is the one given below:

$$u(x) := \sup\{ t \in [a,b] : x \in A_t \}. \quad (4.37)$$

The next result asserts that the function $u$ aforementioned gives rise to continuous function which meets the boundary condition in the strong sense.

**Theorem 4.5.** The function $u$ given in (4.37) satisfies the following

(i) $u = g$ on $\partial \Omega$.

(ii) $u \in C(\bar{\Omega})$.

(iii) $A_t \subset \{ u \geq t \}$ for all $t \in [a,b]$, and $|\{ u \geq t \} \setminus A_t | = 0$ for a.e. $t \in [a,b]$.

The proof of Theorem 3.5 in [80] naturally carries over to prove our Theorem 4.5, since it relies solely on (4.34)-(4.35), plus some basic topological and analytic considerations.

Finally, let us now restate and provide a proof of the main theorem in Chapter 4:

**Theorem 4.1.** For $N \geq 2$, let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with Lipschitz boundary satisfying the barrier condition (4.5), and let $a \in C^3(\bar{\Omega})$ be a non-degenerate weight function, in the sense of (4.2). Then for any boundary data $g \in C(\partial \Omega)$, the function $u$ defined (4.37) is a continuous solution to

$$\inf \left\{ \int_{\Omega} a(x)|Dv| : v \in BV(\Omega), \quad v = g \text{ on } \partial \Omega \right\}, \quad (4.38)$$

where $g \in C(\partial \Omega)$, and $v = g$ is understood in the sense of traces of BV functions. Furthermore, the superlevel sets of this function minimize the weighted perimeter measure $\mathcal{P}_a(\cdot, \Omega)$ with respect to competitors meeting the boundary conditions imposed by $g$ on $\partial \Omega$. 

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Proof of Theorem 4.1. We follow the outline of the proof of Theorem 3.7 in [80]. For any competitor \( v \in BV(\Omega) \) of (4.38), consider the extension \( \bar{v} \in BV(\mathbb{R}^N) \) of \( v \) with \( \bar{v} = G \) in \( \mathbb{R}^N \setminus \bar{\Omega} \), and let \( F_t = \{ \bar{v} \geq t \} \). It is sufficient to show that for a.e. \( t \in [a,b] \),

\[
\mathcal{P}_a(E_t, \Omega) \leq \mathcal{P}_a(F_t, \Omega),
\]

since the weighted co-area formula (cf. Proposition 4.2) would then yield

\[
\int_\Omega a(x) |Du| = \int_a^b \mathcal{P}_a(E_t, \Omega) \, dt \leq \int_{-\infty}^{+\infty} \mathcal{P}_a(F_t, \Omega) \, dt = \int_\Omega a(x) |D\bar{v}| < \infty.
\]

Let us start by noting that \( F_t \setminus \bar{\Omega} = \mathcal{L}_t \setminus \bar{\Omega} \), while for all \( t \in [a,b] \) the set \( E_t \) minimizes the \( a \)-perimeter amongst competitors satisfying this condition. Hence,

\[
\mathcal{P}_a(E_t, \mathbb{R}^N) \leq \mathcal{P}_a(F_t, \mathbb{R}^N),
\]

or equivalently,

\[
\mathcal{P}_a(E_t, \mathbb{R}^N \setminus \bar{\Omega}) + \mathcal{P}_a(E_t, \partial \Omega) + \mathcal{P}_a(E_t, \Omega) \\
\leq \mathcal{P}_a(F_t, \mathbb{R}^N \setminus \bar{\Omega}) + \mathcal{P}_a(F_t, \partial \Omega) + \mathcal{P}_a(F_t, \Omega).
\]

The characterization (4.9) of the \( a \)-perimeter measure and the fact that \( F_t = E_t \) on \( \mathbb{R}^N \setminus \bar{\Omega} \) show that the above inequality reduces to

\[
\mathcal{H}^{N-1} \sqcap a(\partial^* E_t \cap \partial \Omega) + \mathcal{P}_a(E_t, \Omega) \leq \mathcal{H}^{N-1} \sqcap a(\partial^* F_t \cap \partial \Omega) + \mathcal{P}_a(F_t, \Omega). \tag{4.40}
\]

On the other hand, let us note that Lemma 4.1 implies that the set \( \partial^* E_t \cap \partial \Omega \) is \( \mathcal{H}^{N-1} \)-null for a.e. \( t \in [a,b] \). Indeed, \( \{ g = t \} \) is a \( \mathcal{H}^{N-1} \)-null set for all but countably many \( t \in [a,b] \), since \( \mathcal{H}^{N-1}(\partial \Omega) < \infty \). From this and (4.6) we get \( \mathcal{H}^{N-1} \sqcap a(\partial^* E_t \cap \partial \Omega) = 0. \)

Thereby, in light of (4.40), we will have established (4.39) once we prove

\[
\mathcal{H}^{N-1} \sqcap a(\partial^* F_t \cap \partial \Omega) = 0.
\]

This will be argued in the same way as before, once we are able to prove that

\[
\partial_M F_t \cap \partial \Omega \subset \{ g = t \}. \tag{4.41}
\]
Let us recall $g$ is the trace on $\partial \Omega$ of $v$ admissible in (4.38), so for $\mathcal{H}^{N-1}$-a.e. $x \in \partial \Omega$,

$$
\lim_{r \to 0} \int_{B_r(x) \cap \Omega} |v(y) - g(x)| \, dy = 0.
$$

(4.42) (cf. [83, §5.14]). Thus in order to prove (4.41), we consider any $x \in \partial \Omega$ as in (4.42) such that $x \in \{g < t\}$, say $g(x) = t - \delta$ with $\delta > 0$. It follows that

$$
0 = \lim_{r \to 0} \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x) \cap \Omega \cap \{v \geq t\}} |v(y) - g(x)| \, dy
\geq \delta \lim_{r \to 0} \frac{|B_r(x) \cap \Omega \cap \{v \geq t\}|}{|B_r(x) \cap \Omega|}.
$$

Analogously, as $g$ is the trace of $\bar{v} \in BV(\mathbb{R}^N \setminus \Omega)$, the argument above shows

$$
\lim_{r \to 0} \frac{|B_r(x) \cap (\mathbb{R}^N \setminus \Omega) \cap \{\bar{v} \geq t\}|}{|B_r(x) \cap (\mathbb{R}^N \setminus \Omega)|} = 0.
$$

These two identities above imply

$$
\Theta(\{\bar{v} \geq t\}, x) := \limsup_{r \to 0} \frac{|B_r(x) \cap \{\bar{v} \geq t\}|}{|B_r(x)|} = 0,
$$
whence $x \notin \partial M \{\bar{v} \geq t\}$, and so $\{g < t\} \subset \partial \Omega \setminus \partial M F_t$. In a similar fashion, we can argue that $\{g > t\} \subset \partial \Omega \setminus \partial M F_t$ by means of (4.42), in light of

$$
\Theta(\{\bar{v} \geq t\}, x) := \liminf_{r \to 0} \frac{|B_r(x) \cap \{\bar{v} \geq t\}|}{|B_r(x)|} \geq 1 - \Theta(\{\bar{v} < t\}, x) = 1,
$$
which holds for a.e $x \in \partial \Omega$ with $g(x) > t$.

The conclusion is $\{g \neq t\} \subset \partial \Omega \setminus \partial M F_t$, which finishes the proof of Theorem 4.1. \qed
Bibliography


Place of birth: Santiago, Chile. Nationality: Chilean.

Research Interests
Partial Differential Equations, Calculus of Variations, Geometric Analysis.

Education

- Indiana University, Bloomington, Indiana, USA.
  Ph.D in Mathematics (June 2018)
    ◊ Advisor: Peter Sternberg.
    ◊ Dissertation Title: Geometric Problems in the Calculus of Variations.

M.A., Mathematics (May 2014).

- University of Chile, Santiago, Chile.
  B.S., Engineering (Dec 2011)
  Mathematical Engineering Degree (Jul 2012)
    ◊ Advisor: Manuel del Pino.
    ◊ Thesis Title: Construction of entire solutions to the singularly perturbed Allen-Cahn equation in $\mathbb{R}^2$ with spatial inhomogeneity with transitions on noncompact curves”.

Preprints

- A. Zuniga, *Continuity of minimizers to weighted least gradient problems.*
Publications


Research Talks

- Continuity of minimizers to weighted least gradient problems in $\mathbb{R}^n$:
  - Special session on Geometric Analysis, AMS Fall Western Sectional Meeting at University of California, Riverside (Nov 5, 2017).
  - Indiana University PDE Seminar (Sept 23, 2017).

- A geometric approach for the heteroclinic connection problem of gradient systems:
  - Indiana University PDE Seminar (Oct 17, 2016).
  - University of Chile DIM* PDE Seminar (Jul 18, 2016).
  - University of Reading poster session in LMS-CMI Research School† (Jul 6, 2016).

Expository Talks

- An $L^\infty$ constrained least gradient problem in $\mathbb{R}^n$. IU‡ Grad PDE Seminar (Spr 2017).

- A survey on regularity theory of minimal surfaces. IU Grad PDE Seminar (Fa 2016).

- Reduced boundary of Caccioppoli sets. IU Grad PDE Seminar (Fa 2016).

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Service

- Chair of contributed papers session II, AMS sectional meeting at IU (Apr 1, 2016).
- Co-organizer PDE/Analysis Seminar for graduate student at IU (Fa 2016, Spr 2017).
- Volunteer for University of Chile Math Summer School Program: PreCalculus courses aim towards high-school students (Month of Jan: 2008 until 2012).

Awards and Honors

- Hazel King Thompson Fellowship (In support of promising graduate students during dissertation, Math Department IU, Spr 2017).
- Glenn Schober Memorial Travel Award (Math Department IU, Sum 2016).
- David A. Rothrock Award (In recognition of excellence in the teaching of Mathematics, Math Department IU, Spr 2016).
- Becas Chile Study Abroad Doctoral Fellowship (Conferred by the Chilean government to professionals who pursue a graduate degree abroad)
- International Graduate Student Fellowship (Conferred on incoming graduate students having outstanding application, College of Arts and Sciences IU, Fa 2012).
- Best Teaching Assistant Award (Conferred on undergraduate assistants with remarkable teaching evaluations, Department of Mathematical Engineering University of Chile, Fa 2010).
- Outstanding Student Award (Conferred on the top 10% students based on yearly performance, Engineering School of University of Chile, years: 2006, 2007, 2008 & 2010).
- Excellence Scholarship for Freshmen (Conferred on the top ten applicants every year, Engineering School of University of Chile, Fa 2006).
- Maximum score in Physics National Entrance Exam Award (From a pool of nearly two hundred-thousand applicants to undergraduate programs in Chile this award is a recognition
of the twelve students who obtained a perfect score on the Physics entrance test, Chile

Teaching Experience

◊ Department of Mathematics, Indiana University.

Course Instructor for:

• M118 Finite Mathematics (Sum 2017)  • J110 Intro to Problem Solving (Sum 2015)
• M343 Ordinary Diff. Eq (Sum 2016)  • M014 Basic Algebra (Fa 2015)
• D117 Intro to Finite Math II (Spr 2016)  • M025 Pre-Calculus (Spr 2014)

Teaching Assistant for:

• M211 Calculus I (Fa 2014, Spr 2013)  • M118 Finite Math (Fa 2013, Fa 2012)
• M119 Survey of Calculus I (Sum 2013)

Course Grader for:

• M540 PDE I (Spr 2018)  • M544 ODE I (grad*, Fa 2016, Fa 2017)
• M414 Intro. Analysis II (Spr 2018)  • M511 Real Analysis I (grad, Fa 2016)
• M413 Intro. Analysis I (Fa 2017)  • M564 Probability II (grad, Spr 2017)

◊ Department of Mathematical Engineering, University of Chile.

Teaching Assistant for (equivalent in 2015 curriculum)

• MA2002 Vector Calculus (Fa 2012, Sp 2011)  • MA2601 ODE (Fa 2012, Spr 2011, Fa 2011)

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• MA4802 PDE (Spr 2010)  • MA1102 Linear Algebra (Spr 2009)

• MA4002 Variational Differential Calculus (Spr 2010)  • MA1101 Abstract Algebra (Fa 2009)

• MA4801 Functional Analysis (Fa 2010)  • MA1001 Calculus I (Fa 2009)

• MA2001 Calculus III (Fa 2011, Fa 2010)  • FI2004 Statistical Physics (Fa 2008)

Conferences & Summer Schools Attended

• Workshop on Liquid Crystals, Soft-matter Packing, and Active Systems: Material and Biological Applications, IMA‡‡, Minnesota, MN (to occur Jan 16-20 2018).

• Summer School in Calculus of Variations and Nonlinear PDE. University of California Berkeley, Berkeley, CA (May 22-29, 2017).


• AMS Sectional Meeting. Indiana University, Bloomington, IN (Apr 1-2, 2017).

• 78th Midwest PDE Seminar. Loyola University, Chicago, IL (Oct 15-16, 2016).

• Modern Topics in Nonlinear PDE and Geometric Analysis (Summer School). University of Reading, Reading, UK (Jul 4-8, 2016).

• 77th Midwest PDE Seminar. University of Cincinnati, Cincinnati, OH (May 8-9, 2016).

• 6th Symposium on Analysis and PDE. Purdue University, West Lafayette, IN (Jun 1-4, 2015).

• Calculus of Variations and Nonlinear PDE Summer school & Conference. University of Texas at Austin, Austin, TX (May 18-29, 2015).
• Thematic Program on Nonlinear PDEs in Geometry and Physics. University of Notre Dame, Southbend, IN (Summer school & Conference: Jun 9-20, 2014).

• CAPDE$^\dagger$ Seminars on Nonlinear PDEs and Calculus of Variations. Catholic Pontifical University of Chile and University of Chile, Santiago, Chile (Two-year long sequence of talks, 2011-2012.)

• Nonlinear PDE Conference in Valparaiso. UTFSM**, Valparaiso, Chile (Jan 2011).

**Internship Experience**

• Research Assistant - School of Engineering and Sciences, University of Chile, Sep-Nov 2010. “Development of quantitative statistical models to predict performance of freshmen at the School of Engineering”. Supervisor: Nancy Lacourly.


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